Answer Set 6
Physics 240B

A&M 23.2. a) This isn't particularly startling. In the harmonic approximation the low-frequency dispersion relation is linear to leading order. There are multiple branches, with different sound velocities, but the given expression for velocity contains an average of the cubes of the velocities, which is exactly what's needed.

b) In d dimensions the number of states of energy less than ω goes as ω^d, so the density of states goes as ω^{d-1}. (This assumes a linear dispersion relation.)

c) The low-temperature specific heat depends only on the low-frequency part of the dispersion relation, which is linear. $C \propto \frac{d}{dT}\int d\omega D(\omega)\frac{\hbar\omega}{e^{\hbar\omega/kT}-1} \propto \frac{d}{dT}\int d\omega \frac{\omega^d}{e^{\hbar\omega/kT}-1} \propto \frac{d}{dT}(T^{d+1})\int dx \frac{x^d}{e^x-1} \propto T^d$.

d) Going back to part b), if $\omega \propto k^\nu$, then the number of k values with frequency less than ω is proportional to $(\omega^{1/\nu})^d$, so the density of states goes as $\omega^{d/\nu-1}$. In the energy integral, this gives $d/\nu - 1$ temperature factors. Two more come from the $\hbar\omega$ and the $d\omega$. Differentiating with respect to temperature to get specific heat brings the dependence back to $T^{d/\nu}$.

A&M 23.3. a) For one dimension, the number of modes between ω and $\omega + d\omega$ is $\frac{L}{2\pi}dk$, where dk is the length of the range of k's that give frequencies in the right interval. Divide by volume (or length in 1D) to get $g(\omega)d\omega = \frac{dk}{2\pi} = \frac{2d\omega}{2\pi|d\omega/dk|}$. The 2 in the numerator assumes that $\omega(k) = \omega(-k)$. Plugging in the dispersion relation given in the question, $g(\omega) = \frac{1}{\pi\omega_o(a/2)\cos(ka/2)} = \frac{2}{\pi a\sqrt{\omega_o^2-\omega^2}}$.

b) Assume an isotropic maximum at $\mathbf{k} = 0$, $\omega(\mathbf{k}) = \omega_o - Ak^2$. This form of the dispersion relation need only be an approximation that works close to the max. (A max at some other spot would have an offset in $\mathbf{k}$; an anisotropic maximum would have different prefactors for the k_x^2, k_y^2, and k_z^2 terms, and constant-energy surfaces would be ellipsoids rather than spheres. The basic idea is the same.) Consider equation (23.35). Since $\nabla\omega = -2A\mathbf{k}$, we get $|\nabla\omega| \propto \sqrt{\omega_o - \omega}$. The surface integral is over a sphere, with area $4\pi k^2 \propto \omega_o - \omega$. Combining these gives $\sqrt{\omega_o - \omega}$.

1. a) $M\ddot{\mathbf{u}}_{\mathbf{m,n}} = -K_1(2u_{m,n;x} - u_{m+1,n;x} - u_{m-1,n;x})\hat{\mathbf{x}} - K_1(2u_{m,n;y} - u_{m+1,n;y} - u_{m-1,n;y})\hat{\mathbf{y}} - K_2[\sqrt{2}(u_{m,n;x} + u_{m,n;y}) - \frac{u_{m+1,n+1;x}+u_{m+1,n+1;y}+u_{m-1,n-1;x}+u_{m-1,n-1;y}}{\sqrt{2}}]\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}} - K_2[\sqrt{2}(u_{m,n;x} - u_{m,n;y}) - \frac{u_{m+1,n-1;x}-u_{m+1,n-1;y}+u_{m-1,n+1;x}-u_{m-1,n+1;y}}{\sqrt{2}}]\frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}}{\sqrt{2}}$

b) With $K_1 = K_2$, the x-component of the equation of motion from a) becomes $M\ddot{u}_{m,n;x} = -K_1[4u_{m,n;x} - u_{m+1,n;x} - u_{m-1,n;x} - \frac{1}{2}(u_{m+1,n+1;x} + u_{m+1,n+1;y} + u_{m-1,n-1;x} + u_{m-1,n-1;y} + u_{m+1,n-1;x} - u_{m+1,n-1;y} + u_{m-1,n+1;x} - u_{m-1,n+1;y})]$, and similarly in the y direction. For the given wave vector, try $\mathbf{u}_{\mathbf{m',n'}} = \mathbf{A}e^{i(\mathbf{k}\cdot\mathbf{R}-\omega t)} = \mathbf{A}e^{i(\frac{ka}{\sqrt{2}}(m'+n')-\omega t)}$, where the position of the mass is $\mathbf{R} = m'a\hat{\mathbf{x}} + n'a\hat{\mathbf{y}}$. (I am using m', n' to emphasize that the expression for $\mathbf{u}$ holds for *all* masses.) Substituting this into the equations of motion and cancelling $e^{i(\frac{ka}{\sqrt{2}}(m+n)-\omega t)}$ from each term gives $-\omega^2 MA_x = -K_1[A_x(4 - e^{ika/\sqrt{2}} - e^{-ika/\sqrt{2}}) -$

$\frac{1}{2}(A_x e^{-i\sqrt{2}ka} + A_y e^{-i\sqrt{2}ka} + (A_x + A_y)e^{i\sqrt{2}ka})]$, and similarly in the y direction. These look a bit simpler after turning the exponentials into cosines. We can get the solutions by eyeballing the equations; or, alternatively, by trying pure longitudinal and transverse modes. The direction chosen has enough symmetry that the modes are longitudinal and transverse. The longitudinal mode, with $A_x = A_y$, has identical equations in the x and y directions, $M\omega^2 = K_1(4 - 2\cos\frac{ka}{\sqrt{2}} - 2\cos\sqrt{2}ka) = K_1(4\sin^2\frac{ka}{2\sqrt{2}} + 4\sin^2\sqrt{2}ka)$, which define ω. For the transverse mode, $A_x = -A_y$ and ω is defined by $M\omega^2 = K_1(4 - 2\cos\frac{ka}{\sqrt{2}} - 2) = 4K_1\sin^2\frac{ka}{2\sqrt{2}}$.

c) Keep same wave vector, $\mathbf{k} = k(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, but change the ratio of spring constants. The system still has symmetry in the propagation ($\mathbf{k}$) direction, and again the modes end up being longitudinal and transverse. The former has frequency obeying $M\omega^2 = K_1(2 - 2\cos\frac{ka}{\sqrt{2}}) + \frac{1}{4}K_1(2 - 2\cos\sqrt{2}ka) = K_1(4\sin^2\frac{ka}{2\sqrt{2}} + \sin^2\frac{ka}{\sqrt{2}})$, the latter $M\omega^2 = 4K_1\sin^2\frac{ka}{2\sqrt{2}}$.

d) For polarization in the x direction, set all y-components of $\mathbf{u}_{m,n}, \mathbf{u}_{m+1,n}$, etc. to zero. The equation of motion for the y direction becomes $0 = M\ddot{u}_{m,n;y} = \frac{1}{4}K_1(u_{m+1,n+1;x} + u_{m-1,n-1;x} + u_{m+1,n-1;x} + u_{m-1,n+1;x})$. As usual assume a plane wave in some direction $\mathbf{k}$. Substituting this in the EOM and using an angle addition formula gives $0 = 2\cos(k_x a + k_y a) + 2\cos(k_x a - k_y a) = 4\cos k_x a \cos k_y a$, which only vanishes when one of k_x, k_y equals $\pi/2a$. I probably should have stated the question less strongly. There do exist a few very special wave vectors that support x polarization, but most $\mathbf{k}$ do not. This implies that for $\mathbf{k} = k\hat{\mathbf{x}}$, most magnitudes of k do not yield a perfectly longitudinal wave.

2. a) In a Debye model each branch of the dispersion curve is linear, and the modes cover a sphere which has the same volume as the first Brillouin zone. The zero-point energy per volume is $\frac{1}{V}\sum_{modes}\frac{1}{2}\hbar\omega(k) \approx 3(\frac{1}{8\pi^3})\int_0^{k_D} 4\pi k^2 dk(\frac{1}{2}\hbar ck) = \frac{3\hbar c}{16\pi^2}k_D^4 = \frac{3}{16\pi^2}(6\pi^2 n)\hbar ck_D = \frac{9}{8}n\hbar\omega_D$.

b) In an Einstein model, all modes have the same frequency ω_o, independent of k, so the total zero-point energy is $(\frac{1}{2}\hbar\omega_o)(3n)$ per volume.

3. The roughly linear portion on the log-log plot indicates a power law dependence of C_V on T. The slope of 3 shows the dependence is in fact cubic. (When I say "slope" here I am thinking of the x and y axes as labelled by the exponents—that is, that the quantities plotted are truly the logarithms.) $C_V = \beta T^3$ gives $\log C_V = \log\beta + 3\log T$, so the y-intercept identifies the prefactor. In the figure the intercept is the log of C_V at $T = 10^0$ K, or about -8. Thus $10^{-8}(4.184 \times 10^7)\text{erg} \approx \beta = \frac{12N\pi^4}{5}k_B(1/\Theta_D)^3$. Here N is the number of atoms in one gram of diamond, $6.02 \times 10^{23}/12$. I get $\Theta_D \approx 1570$ K.