CDT and $2+1$ Dimensional Hořava-Lifshitz Gravity on a Lattice

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Abstract

We extend the discrete Regge action used in current CDT Monte Carlo simulations to include anisotropic terms inspired by the Hořava-Lifshitz action in $2+1$ dimensions. We demonstrate the existence of extended phases of geometry which agree with the behavior of solutions to the classical equations of motion. We sketch the overall layout of the phase diagram and initiate a study of the behavior of the model near phase transitions.

1 Introduction

The Causal Dynamical Triangulations (CDT) program attempts to define quantum gravity in terms of a statistical system of dynamical geometry. Considered a conservative approach to answering questions of quantum gravity, CDT appeals only to the same tools applied with great success to other local quantum field theories [7]. In essence, the program defines the path integral

$$\int_{\mathcal{G}=\frac{\mathtt{Lor(M)}}{\mathtt{Diff(M)}}} [\mathcal{D}g_{\mu\nu}] e^{iS[g_{\mu\nu}]} \tag{1.1}$$

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in terms of the partition function

$$Z = \sum_T \frac{1}{C_T} e^{iS[T]} \tag{1.2}$$

where the sum is take over simplicial manifolds T constructed from a set of D dimensional simplices and $S[T]$ is a discrete action. The factor $\frac{1}{C_T}$, equal to the inverse of the size of the automorphism group of T, corresponds to the path integral measure. In addition to being a simplicial manifold, T must also have a causal structure. This causal structure amounts to gluing the simplices in a way that preserves a notion of global time. This distinguishes the CDT program from the Euclidean DT program which failed to produce extended phases of geometry in dimensions higher than two [7], [22], [2]. The discrete action $S[T]$ is the discretized Einstein-Hilbert action referred to in this paper as the Regge action. In two $2+1$ dimensions, the case of interest for this paper, the Regge action reduces to a functional of the form $\kappa_3 N_3 - \kappa_0 N_0$ after application of the Dehn-Sommerville relations [18], [5]. The constants κ_0, κ_3 contain the bare Newton constant and cosmological constant and N_i is the total number of simplices of dimension i. We will have more to say about the details of CDT in Section 3.

The structure of the CDT program allows for the use of numerical lattice techniques to probe its properties. Monte Carlo simulations have provided considerable support for the existence of extended phases of geometry [7], [2], [5], [3]. The average geometry observed matches well with the expected (Euclidean) de Sitter behavior [5], [3]. Furthermore, studies done on the ensemble average of the spectral dimension in $3+1$ CDT have revealed not only agreement with classical de Sitter geometry but have also shown that the small scale geometry is effectively two dimensional [6], [8], [7], [2]. This is consistent with the predictions of other quantum field-theoretic approaches to quantum gravity, namely the Hořava-Lifshitz theory [15] and Asymptotically Safe Gravity [21].

Given the success of CDT with the Regge action based on Einstein gravity, it would be fascinating to consider some other gravitational theory amenable to CDT Monte Carlo simulations. In [14], Hořava proposes a different approach to quantum gravity which also relies on the tools of local quantum field theory. The notion of (passive) diffeomorphism invariance is abandoned in the UV to improve the behavior of the graviton propagator in that regime. The diffeomorphism invariance is assumed to be restored as an accidental symmetry as the RG flow takes us into the IR. This approach was largely inspired by anisotropic condensed matter systems and we will sometimes also refer to the Hořava proposal as anisotropic gravity. As we will demonstrate, Hořava-Lifshitz (HL) theory may be adapted for use in CDT Monte Carlo simulations and produces extended phases in $2+1$ dimensions.

We make rather modest goals in using the formalism of HL theory as a testing ground for CDT techniques. It should be noted that interesting and deep connections between the two seemingly disparate approaches have been observed [2], [1], [16]. Moreover, there is the tantalizing prospect of using the HL discrete action in CDT lattice simulations to discern qualitative information about the quantum nature of anisotropic gravity. We will leave such speculations for Section 4.

The outline of the paper is as follows. In Section 2, we will present a lightning introduction to Hořava-Lifshitz theory, highlighting the salient features needed for our presentation. References for further study of this large and exciting field are provided. In Section 3, we outline the underlying notions of CDT and discuss in detail the construction of the discrete action in $2+1$ dimensions. In Section **??** we present our main results: the presence of extended geometries and the correspondence with classically derived spacetimes, the layout of the phase diagram and the behavior of a suitably chosen order parameter near a phase transition. Section 4 will summarize our conclusions and discuss future work that will be of interest to the HL community. In order to streamline the presentation, Appendix A contains the construction of the solutions to the equations of motion.

2 Basics of Hořava-Lifshitz Gravity

HL gravity is predicated on the loss of the diffeomorphism invariance of Einstein gravity in the UV and the restoration of this symmetry in the IR under the RG flow of couplings. The interested reader may consult [14], [13], [16] for the genesis of the proposal and the excellent general overviews [26], [29]. See [23] for a review of the cosmological implications of anisotropic gravity. In this paper, we will not concern ourselves with the physics implications of HL gravity. As emphasized in the introduction and reiterated here, testing CDT ideas against a gravitational theory fundamentally different from Einstein gravity will be our primary objective.

The configuration space of HL gravity in D spatial dimensions consists of foliated manifolds with topology $M = \Sigma \times I$. For simplicity, we will assume that Σ has the topology of a D dimensional sphere. The foliated structure of the manifold reflects the loss of invariance under Diff(M). The symmetry group is instead $\mathrm{Diff}_F(M)$, the group of foliation preserving diffeomorphisms. Concretely, if we introduce a local smooth coordinate chart (x^i, t), $\mathrm{Diff}_F(M)$ consists of reparametrizations of the form

$$\tilde{x}^i = \zeta^i(x,t), \tilde{t} = f(t) \tag{2.1}$$

Note that, though spatial diffeomorphisms may be time-dependent, reparametrizations of time must be independent of space. One may naturally wonder what advantage is gained by such a radical shift in thinking. It is well known that including higher curvature terms in the gravitational action can make the field theory renormalizable [10], [28]. However, such theories suffer from loss of unitarity and propagation of ghosts [25]. These issues stem from the inclusion of higher time derivatives in addition to higher spatial derivatives in the action thus maintaining the democracy of space and time.

However, by taking the symmetries to be foliation preserving only, we are allowed to write down an action quadratic in time derivatives while simultaneously including higher spatial derivatives. The anisotropic theory obtained this way is power-counting renormalizable [14], [16]. Furthermore, the couplings of the higher spatial derivative terms are expected to flow to their relativistic values under the RG in the infrared. This restores the full diffeomorphism symmetry inherent to Einstein gravity and essential for phenomenology as an accidental symmetry.

The ADM formalism is the most natural to use in light of the foliated structure of the underlying manifold. The field content consists of the spatial metric tensor $g_{ij}(x,t)$, the shift vector $N^j(x,t)$ and the lapse function $N(t)$. Note that we are restricting the lapse to be a function only of time. This is the so-called projectable version of the theory. The reader interested in the more important aspects of the projectable version is referred to [26], [29], [30]. See also [9] for a treatment of the projectable case and the so-called healthy extension of HL theory. We motivate the use of the projectable case in this context as follows: the time-like links of the simplices used in CDT have fixed length and this should be reflected in the behavior of the lapse. We will have more to say about this soon.

Since we wish to maintain an action quadratic in time-derivatives, the kinetic piece must be constructed out of invariants involving $\dot{g}_{ij}$. This quantity is clearly not covariant under the symmetry group $\mathrm{Diff}_F(M)$. However, the extrinsic curvature tensor

$$K_{ij} = \frac{1}{2N(t)} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_i N_j\right) \tag{2.2}$$

satisfies this criterion. The covariant derivative ∇ is constructed from the metric tensor on the spatial slice. The most general kinetic piece compatible with $\mathrm{Diff}_F(M)$ is given by

$$\frac{2}{\kappa^2}\int_M dtd^Dx\sqrt{g}N\left(K_{ij}K^{ij}-\lambda K^2\right) \tag{2.3}$$

where κ^2 plays the role of Newton's constant. The parameter λ arises from the de Witt super metric [14], [13] and is free to flow under the RG. The full diffeomorphism invariance of Einstein gravity fixes this parameter to $\lambda = 1$.

The potential term of the action is constructed from the spatial metric tensor and its spatial derivatives. In order to guarantee power-counting renormalizability of the action, the potential piece will include terms of order $2D$.[1] If $V[g_{ij}]$ is a scalar functional of the metric tensor, the most general potential piece is

$$\frac{2}{\kappa^2}\int_M \sqrt{g}NV[g_{ij}] \tag{2.4}$$

Putting 2.3 and 2.4 together, the HL action is

$$\frac{2}{\kappa^2}\int_M dtd^Dx\sqrt{g}N\left(K_{ij}K^{ij}-\lambda K^2-V[g_{ij}]\right) \tag{2.5}$$

It may strike the reader that there is a proliferation of terms for dimensions D larger than 2. This need not concern us since we are considering only $D = 2$. The most general anisotropic action in this case is

$$\frac{2}{\kappa^2}\int_M dtd^2x\sqrt{g}N\left(K_{ij}K^{ij}-\lambda K^2-\alpha R^2+\beta R-2\Lambda\right) \tag{2.6}$$

where R is the Ricci scalar of the spatial metric tensor. In this paper, we will refer to α as

[1]For those familiar with the parlance of anisotropic gravity, the dynamical critical exponent z equals $2D$

the R^2 coupling and λ as the K^2 coupling for ease of exposition. Λ will be referred to as the cosmological constant in keeping with tradition.

A glance at the equation of motion (A.3) derived by varying the spatial metric in (2.6) reveals the absence of the β coupling. Indeed, the R term is a total derivative and the integral over M depends only on the topology of the spatial slice and the time extension.[2] This follows immediately from the Gauss-Bonnet theorem.

It is important to note here that the β coupling does influence the classical equations of motion derived from the full projectable version of HL gravity. Variation of the lapse will produce the non-local integral constraint

$$\frac{2}{\kappa^2}\int_\Sigma d^2x\sqrt{g}\left(K_{ij}K^{ij} - \lambda K^2 + \alpha R^2 - \beta R + 2\Lambda\right) = 0 \tag{2.7}$$

As mentioned in [16] and demonstrated in [17], this constraint restricts the form of spatially homogeneous and isotropic FRW solutions. Therefore, the classical phase diagram appearing in [16] and [17] will be drastically different from the classical phase diagram obtained in the present setting.

When (2.6) is discretized in Section 3 for use as a functional on CDTs, we will see that the lapse is fixed to be time-independent in the process. This is due to the geometric restrictions imposed by using simplices with fixed link length in constructing the triangulated manifold. In effect, we are considering a reduced version of projectable HL in which (2.7) is ignored [3]. Since we are interested only in using HL gravity as a test-bed for CDT ideas, we will only be concerned in demonstrating the existence of extended phases of geometry that match the classical solutions to the reduced equations of motion found in Appendix A; i.e. those

[2] In Section 3, it is noted that the spacetimes comprising the ensemble have periodic boundary conditions in the time direction. The time extension is held fixed and the β term in total contributes only an additive constant to the action.

[3] See [9] for a discussion of the physical content of this reduced version of HL theory

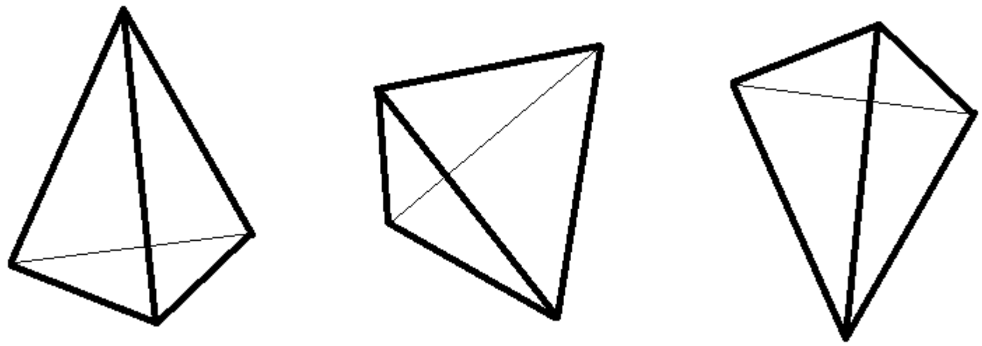

Figure 1: From right to left we have the (3,1), (2,2) and (1,3) tetrahedra used to construct Causal Dynamical Triangulations (CDTs)

obtained by varying the metric and the shift vector, but not the lapse function.

3 The Discrete Hořava-Lifshitz Action

The $2+1$ dimensional triangulated manifolds comprising the statistical ensemble of CDT geometries are constructed from three basic tetrahedra. The triangulated manifolds are required to have a causal structure and a global notion of time. This causal structure allows for a well-defined notion of Wick rotation and is an essential ingredient for curing the pathologies presented in the forerunner Euclidean DT program [2], [5].[4] The causal structure is enforced by triangulating the spatial slices (here having spherical topology) with spatial triangles and connecting the vertices of these triangles with time-like links. The vertices in adjacent spatial slices are connected in such a way as to produce only the $(3,1)$ the $(1,3)$

[4]Forbidding topology change (i.e. the formation of baby universes) is the other essential ingredient.

and the $(2,2)$ tetrahedra of Figure 1. The lengths of the spatial links are fixed to be $l_s = a$ providing a short distance or UV cutoff. In Lorentzian signature, the lengths of space-like and time-like links are related by

$$\frac{l_t^2}{l_s^2} = -\eta < 0 \tag{3.1}$$

Note that our parameter η corresponds to the parameter α common to CDT literature; we reserve α for the R^2 coupling constant.

The partition function (1.2) is evaluated with the Regge action $S[T] = \kappa_3 N_3 - \kappa_0 N_0$ where N_0 is the number of vertices and N_3 is the number of tetrahedra in the CDT T. The constants κ_0 and κ_3 contain the bare coupling constants. These quantities are all evaluated in Lorentzian signature and then Wick rotated into Euclidean signature to perform numerical lattice simulations as in [18]. The simulations themselves are performed using Monte Carlo methods and the Metropolis algorithm. The details of the Metropolis algorithm, relevant to both traditional CDT simulations and the extension considered here, may be found in [18].

As in the Euclidean DT program, an unphysical phase in $2+1$ dimensions is observed [5]. In this phase, the time slices are effectively decoupled from each other. However, beyond a certain critical value for the cosmological constant, an extended phase is observed. The ensemble average geometry in this regime matches Euclidean de Sitter space. In $3+1$ dimensions, the phase diagram has a richer structure though an extended phase with the appropriate average geometry appears [3], [2], [7].[5]

In the following sections we will present our arguments for constructing a discrete analogue of the continuum HL action (2.6) appropriate for CDT Monte Carlo simulations. The ultimate objective will be to produce extended phases as witnessed in Regge action-based CDT.

[5] We will make a few comments regarding this phase diagram in Section 4 given the tantalizing connection between it and the phase diagram appearing in HL gravity in $3+1$ dimensions [1].

3.1 An Anisotropic CDT Action

It is our assertion that two very reasonable criteria must be satisfied when placing HL gravity on the lattice for the sake of simplicity and consistency:

1. The discrete action should reduce manifestly to the discrete Regge action when all couplings are dialed to their relativistic values; i.e. $\lambda = 1$ etc.

2. The transfer matrix defined on the space of boundary geometries must yield a well-defined Hamiltonian.

To see how we might satisfy the first condition, let us write the continuum action (2.6) as

$$S_{EH}[g_{\mu\nu}] + \frac{2}{\kappa^2}(1-\lambda)\int_M dtd^2x\sqrt{g}N(t)K^2 - \frac{2}{\kappa^2}\alpha\int_M dtd^2x\sqrt{g}N(t)R^2 \tag{3.2}$$

This rearrangement is possible due to the Gauss-Codazzi relations. The additional terms represent anisotropic contributions. The advantage of this formulation is that we may manifestly reproduce the continuum Einstein-Hilbert action by dialing the couplings to their relativistic values. This trivial rewriting instructs us on how to formulate the discrete HL action: we simply need discrete analogues of the R^2 and K^2 integrals.

For clarity, we will discuss first the discrete analogue of the R^2 term. This will allow us to introduce notions relevant to both anisotropic terms while circumventing the additional challenges that criterion (2) produces in building the analogue K^2 term.

3.1.1 The R^2 Term

In the Regge calculus, the Ricci scalar is defined in terms of deficit angles about $d-2$ dimensional hinges if the simplicial complex has total dimension d [24]. However, curvature is defined in a distributional sense and is concentrated at the hinges. It is therefore unclear what meaning can be given to curvature squared terms in this context. In what follows we will adhere to the philosophy of [11] and [4] when considering discrete analogues of curvature squared terms. As stated in [4], it is not the task of the CDT program to approximate a given continuum manifold and so we will not be overly concerned with the distributional definition of the Ricci scalar. Following the lead of [4], we fix a spatial slice and assign an area A_v to each vertex. This area is viewed as an appropriate share of the area of all spatial triangles containing the vertex v . Specifically, if $N_\triangle(v)$ represents the number of spatial triangles sharing v in its spatial slice, then

$$A_v = \frac{1}{6} \sum_{\triangle \ni v} A_\triangle = \frac{1}{6} \frac{\sqrt{3}}{4} a^2 N_\triangle(v) \tag{3.3}$$

since the triangles in the spatial slices are assumed to be equilateral. We define the curvature density at the vertex v as

$$R = \frac{\delta_v}{A_v} \equiv \frac{2\pi - \frac{\pi}{3} N_\triangle(v)}{\frac{1}{6}\frac{\sqrt{3}}{4} a^2 N_\triangle(v)} \tag{3.4}$$

The numerator δ_v is recognized as the deficit angle about the vertex v through shared triangles in the spatial slice. Modulo uninteresting numerical factors which may be absorbed into the definition of α, the discrete analogue of $\int_{\Sigma_t} d^2x \sqrt{g} R^2$ is

$$\frac{1}{a^2} \sum_{v \in V_t(T)} N_\triangle(v) \left(\frac{6 - N_\triangle(v)}{N_\triangle(v)} \right)^2 \tag{3.5}$$

with $V_t(T)$ denoting the set of vertices belonging to the time t spatial slice of the CDT T. The most natural discretization of the time integral is

$$\int dt N(t) \to \sum_t \sqrt{\eta} a \tag{3.6}$$

[6] and so we make the identification

$$\int_M dtd^2x\sqrt{g}N(t)R^2 \to \frac{\sqrt{\eta}}{a}\sum_{V(T)}\frac{(6-N_{\triangle}(v))^2}{N_{\triangle}(v)} \tag{3.7}$$

where $V(T)$ denotes the vertex set of T.

3.1.2 The K^2 Term

Defining the discrete version of the K^2 term appearing in the 2+1 dimensional action follows the same reasoning as in the previous section but with some additional complications. If we adhere to the prescription in [4], then schematically we should have

$$\int_M dtd^2x\sqrt{g}NK^2 \to \sum_t \sum_{o\in O_t(T)} V_s(o)\left(\frac{\delta(o)V(o)}{V_s(o)}\right)^2 \tag{3.8}$$

where o is the object assigned the extrinsic curvature density, $V(o)$ is the volume of the object and $V_s(o)$ is the shared-volume of o; i.e. the volume of all top-dimensional objects containing o. We are being purposefully abstract to motivate the definition used in our simulations. In the R^2 case, the choices of object type o and top-dimensional objects contributing to the share-volume of o were quite clear given the intrinsic nature of the Ricci scalar. Put another way, since the Ricci scalar is part of the intrinsic geometry of the continuum hypersurface, there is sufficient reason to choose the o object type as a vertex and the top-dimensional objects as spatial triangles.

The choice is not as clear cut in the case of extrinsic curvature which is sensitive (in the continuum) to how the hypersurface is embedded in the ambient manifold. The discrete definition of the K^2 term must preserve as many of the geometric properties of its continuum

[6] As mentioned in Section 2, this discretization of the time integral assumes a constant lapse gauge during the construction of the action. It is this assumption that rules out the non-local integral constraint (2.7)

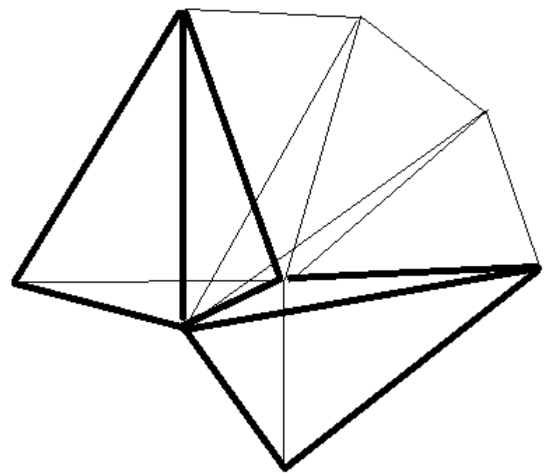

Figure 2: An embedding in three dimensions of two (3,1) tetrahedra (solid black) joined by three (2,2) tetrahedra (thin black) all sharing a common edge. A vector perpendicular to the triangular base of a (3,1) simplex will be rotated by an angle $\pi - 2\theta^{(3,1)} - 3\theta^{(2,2)}$ as it is parallel transported across the edge.

counterpart as possible. This observation requires us to take the top-dimensional objects contributing to the share-volume $V_s(o)$ to be tetrahedra. Furthermore, in the continuum, K^2 scales as L^{-2} for the unit of length L . Taken together, these observations imply that the volume $V(o)$ should scale as a^2 and so o should be a spatial triangle.

Based on geometric considerations, we argued that o should be taken to be a spatial triangle. The question of how to assign a deficit angle $\delta(o)$ to a spatial triangle remains. The following is largely inspired by [19], the ideas therein adapted for use in CDT framework. Consider a spatial slice of the CDT T. For each spatial triangle, we may define a future-directed normal vector at the barycenter of its corresponding $(3,1)$ tetrahedron. A natural measure of extrinsic curvature at the interface between two adjacent triangles e would be the deficit angle of the normal vector as it is parallel transported from its initial $(3,1)$ tetrahedron to the other $(3,1)$ tetrahedron containing edge e. The example in Figure 2 can be generalized to show that the deficit angle is

$$\delta_e = \left(\pi - 2\theta^{(3,1)} - \theta^{(2,2)} N^{\uparrow}_{(2,2)}(e)\right) \tag{3.9}$$

with $\theta^{(3,1)}$ and $\theta^{(2,2)}$ being the (Lorentzian) dihedral angles about space-like edges. $N^{\uparrow}_{(2,2)}(e)$ represents the number of $(2,2)$ tetrahedra attached to edge e and in the immediate future of e. It is now straightforward to assign extrinsic curvature to the spatial triangle $\triangle$ as

$$\delta_\triangle = \left(3\pi - 6\theta^{(3,1)} - \theta^{(2,2)} N^{\uparrow}_{(2,2)}(\triangle)\right) \tag{3.10}$$

which is reminiscent of a trace.

To summarize, a logical first choice for the discrete K^2 term could be

$$\int_M dtd^2x\sqrt{g}NK^2 \to a^4 \sum_{\triangle \in N_2^{\texttt{spatial}}(T)} \frac{\left(3\pi - 6\theta^{(3,1)} - \theta^{(2,2)} N^{\uparrow}_{(2,2)}(\triangle)\right)^2}{4V^{(3,1)} + V^{(2,2)} N^{\uparrow}_{(2,2)}(\triangle)} \tag{3.11}$$

The denominator is the total (Lorentzian) volume of the tetrahedra that share $\triangle$ and are to its immediate future; i.e. the share-volume.

There is a subtlety heretofore overlooked regarding criteria (2). Following the construction of [18], if we wish to ensure the existence of a well-defined Hamiltonian on the space of boundary geometries, it becomes necessary to symmetrize the above term in the time direction. The square of the transfer matrix yields such a Hamiltonian if it can be proven that the transfer matrix is symmetric and satisfies the Osterwalder-Schrader positivity condition exactly as shown in [18]. Obviously, the action based on our postulated discrete K^2 term would fail to satisfy these criteria given the clear asymmetric time dependence. A simple fix would be to symmetrize the K^2 term in the time direction. A bit of thought should convince the reader that no essential geometric content is changed by doing so.

We conclude this section by summarizing the above conclusions with

$$\int_M dtd^2x\sqrt{g}NK^2 \to a^4 \sum_{\triangle\in N_2^{\mathtt{spatial}}(T),b=\uparrow,\downarrow} \frac{\left(3\pi - 6\theta^{(3,1)} - \theta^{(2,2)}N^b_{(2,2)}(\triangle)\right)^2}{4V^{(3,1)} + V^{(2,2)}N^b_{(2,2)}(\triangle)} \tag{3.12}$$

with obvious notations and modulo uninteresting numerical coefficients.

3.1.3 The Euclidean Action

In this section, we piece together the total discrete HL action and perform a Wick rotation. In Lorentzian signature [18], the dihedral angles are given by

$$\theta^{(3,1)} = \frac{\pi}{2} + i\log\left(\frac{1+2\sqrt{3\eta+1}}{\sqrt{3}\sqrt{4\eta+1}}\right) \tag{3.13}$$

$$\theta^{(2,2)} = i\log\left(\frac{4\eta+3-2\sqrt{2}\sqrt{2\eta+1}}{4\eta+1}\right) \tag{3.14}$$

The Wick rotation consists of rotating η through the lower complex plane to $-\eta$ [18]. As a consequence, if the argument of an expression of the form $\sqrt{\eta+c}$ becomes negative as a result of the Wick rotation, we interpret it as $-i\sqrt{-(\eta+c)}$. A simple calculation shows that after the Wick rotation

$$\theta^{(3,1)} = \frac{\pi}{2} - \arccos\left(\frac{2\sqrt{3\eta-1}}{\sqrt{3}\sqrt{4\eta-1}}\right)\ ,\ \theta^{(2,2)} = \arccos\left(\frac{4\eta-3}{4\eta-1}\right) \tag{3.15}$$

Also, the Lorentzian volumes

$$V^{(3,1)} = \frac{1}{12}\sqrt{3\eta+1}a^3\ ,\ V^{(2,2)} = \frac{1}{6\sqrt{2}}\sqrt{2\eta+1}a^3 \tag{3.16}$$

become

$$V^{(3,1)} = -i\frac{1}{12}\sqrt{3\eta-1}a^3, V^{(2,2)} = -i\frac{1}{6\sqrt{2}}\sqrt{2\eta-1}a^3 \tag{3.17}$$

after Wick rotation.

The Euclidean discrete HL action is equal to

$$\frac{2}{\kappa^2}(1-\lambda)a \sum_{\triangle\in N_2^{\mathtt{spatial}}(T),b=\uparrow,\downarrow} \frac{\left(3\pi - 6\theta^{(3,1)} - \theta^{(2,2)}N^b_{(2,2)}(\triangle)\right)^2}{2\sqrt{2}\sqrt{3\eta-1}+\sqrt{2\eta-1}N^b_{(2,2)}(\triangle)} +$$
$$\kappa_3(\kappa,\Lambda)N_3 - \kappa_0(\kappa,\Lambda)N_0 + \frac{2}{\kappa^2}\alpha\frac{\sqrt{\eta}}{a}\sum_{V(T)}\frac{(6-N_\triangle(v))^2}{N_\triangle(v)} \tag{3.18}$$

This action is used in the Monte Carlo simulations which produced the results of the next section.

4 Conclusions and Outlook

We have constructed a discrete analogue to the Hořava-Lifshitz action in 2+1 dimensions and used it in CDT Monte Carlo simulations. There is significant evidence that extended phases of geometry are present and match the classical solutions obtained through the continuum action. This supports our central assertion that Causal Dynamical Triangulations is a robust setting to address questions of a quantum nature for anisotropic gravity models as well as Einstein gravity.

The focus in this article has been on the CDT program itself. However, we cannot resist highlighting several possible applications of this work beyond providing a new test-bed for CDT.

- In $2+1$ dimensions, HL gravity has a single propagating scalar degree of freedom which is in stark contrast with Einstein gravity in the same number of spacetime dimensions. This propagating scalar mode has generated much controversy regarding the phenomenological viability of anisotropic gravity [26], [29]. As observed in [27], $2+1$ HL gravity provides a unique window into the dynamics of the scalar mode without introducing the complications of having propagating tensor modes. Studying the quantum dynamics of the scalar mode in the context of CDT simulations would be of great help in understanding the role of the scalar in $2+1$ and ultimately $3+1$ dimensions.

- Perturbations of projectable HL gravity about flat Minkowski space in $3+1$ dimensions have been shown to generate instabilities [20]. The situation is much improved when the background spacetime is de Sitter [30]. There, it is possible to choose the mass scale of higher derivative operators in conjunction with the exponential expansion of space in order to suppress the instabilities of all modes. However, in the relativistic limit $\lambda \to 1$, the higher derivative terms becomes relevant and the linearized analysis breaks down. It has been shown that the non-perturbative Vainshtein mechanism allows one to take the relativistic limit continuously [23], [30]. It would be interesting to understand how this mechanism works in the context of anisotropic CDT simulations.

- The phase diagram of $3+1$ dimensional CDT simulations using the Regge action of Einstein gravity has a structure strikingly similar to that seen in Hořava-Lifshitz gravity [1], [16]. Given that a fixed causal structure is a prerequisite to both approaches to quantum gravity and both predict the same dimensional reduction in $2+1$ and $3+1$, there is well-founded speculation that the two approaches are fundamentally related. In [2], it is even speculated that the tri-critical point in the CDT phase diagram may correspond to an anisotropic continuum limit. It is possible that a continuation of this work to $3+1$ dimensions may provide evidence to support these assertions.

- A recent proposal for a coarse-graining procedure for CDT has appeared in [12]. This opens up the possibility of studying the RG flow of couplings in anisotropic CDT. It is often argued that the couplings of HL gravity will flow to their relativistic values in the IR, thus restoring diffeomorphism invariance as an accidental symmetry. However, very little proof of this assertion exists. Applying the coarse-graining procedure of [12] may yield qualitative information on the flow of couplings for HL gravity. Conversely, one may consider the qualitative results of such an RG study as confirmation of the validity of the coarse-graining procedure based on our expectations from HL gravity as a quantum field theory.

Generalizations of this work include modification of the framework to recover the full projectable HL theory and placing $3+1$ dimensional HL theory on the lattice. The latter would prove challenging due to the proliferation of possible terms in the potential. Furthermore, a discrete action in $3+1$ dimensions would necessarily have terms involving the Riemann tensor that have not been considered in this work. Since we limited our scope to two spatial dimensions, we had only to consider R^2 and K^2 terms. It is perhaps non-trivial to construct discrete analogues of curvature terms that appear in $3+1$ dimensional HL gravity.

The former may be possible if we exploit the observation in [9], [27] that the Stückelberg formalism allows us to rewrite Hořava-Lifshitz theory in terms of general relativity coupled to a scalar field referred to as the khronon. We hope to return to this possibility in the near future.

Much work remains. The cross-fertilization of CDT and Hořava-Lifshitz gravity proves to be a fruitful area of research and promises to yield new insights into two leading contenders for a quantum theory of gravity.

A Appendix: The Classical Equations of Motion

This appendix collects the classical equations of motion and the solutions relevant to the Monte Carlo simulations performed for this paper. We will assume a Euclidean signature and a compact spatial slice with the topology of S^2.The action

$$S = \frac{2}{\kappa^2} \int_M dt d^2x \sqrt{g} N(t) \left(\lambda K^2 - K_{ij} K^{ij} - \alpha R^2 + \beta R - 2\Lambda \right) \tag{A.1}$$

implies the equations of motion

$$\nabla_i \pi^{ij} = 0 \tag{A.2}$$

$$\begin{aligned} -\frac{1}{\sqrt{g} N(t)} \partial_t \left(N(t) \sqrt{g} \pi^{ij} \right) + \frac{1}{2} g^{ij} \left(K_{kl} K^{kl} - \lambda K^2 - \alpha R^2 + 2\Lambda \right) - 2K^{il} K_l^j + 2\lambda K K^{ij} \\ + 2\alpha \nabla^i \nabla^j R - 2\alpha \nabla^2 R g^{ij} + \frac{1}{2} \left(\nabla_l N^i \pi^{jl} + \nabla_l N^j \pi^{il} - \pi^{ij} \nabla_l N^l \right) = 0 \end{aligned} \tag{A.3}$$

where $\pi_{ij} = K_{ij} - \lambda K g_{ij}$. Equation (A.2) is obtained by varying the shift vector and is referred to as the momentum constraint. Equations (A.3) come about by varying the spatial metric and will be referred to as the metric equations of motion. We exclude the non-local integral constraint that would arise from variations of the lapse.[7]

We consider a spatially homogeneous and isotropic metric ansatz of the form $g_{ij} = a(t)^2 \hat{g}_{ij}$ with $\hat{g}_{ij}$ the round sphere metric. The shift vector N_j is taken to be identically 0. With this assumption, the momentum constraint (A.2) is trivially satisfied.

[7]See Section 2

A.1 Case $\lambda = 1, \Lambda > 0$

In this case, the metric equations of motion are satisfied if and only if

$$\frac{\ddot{a}}{a} = \frac{\alpha}{2a^4} - \Lambda \tag{A.4}$$

This nonlinear equation in $a(t)$ implies that

$$\left(\frac{d}{dt}\left(a^2 - \frac{C}{2\Lambda}\right)\right)^2 = -2\alpha + \frac{C^2}{\Lambda} - 4\Lambda\left(a^2 - \frac{C}{2\Lambda}\right)^2 \tag{A.5}$$

where C is a constant of integration. If we write $u(t) = a(t)^2 - \frac{C}{2\Lambda}$, then (A.5) can be written as

$$(\dot{u})^2 + 4\Lambda u^2 = -2\alpha + \frac{C^2}{\Lambda} \tag{A.6}$$

which has the solution

$$u(t) = \sqrt{-\frac{\alpha}{2\Lambda} + \frac{C^2}{4\Lambda^2}} \cos\left(2\sqrt{\Lambda}t + \delta\right) \tag{A.7}$$

or

$$a(t)^2 = \frac{C}{2\Lambda} + \sqrt{-\frac{\alpha}{2\Lambda} + \frac{C^2}{4\Lambda^2}} \cos\left(2\sqrt{\Lambda}t + \delta\right) \tag{A.8}$$

for positive values of α.

When the sign of α is negative, we have the solution

$$a(t)^2 = \frac{C}{2\Lambda} + \sqrt{\frac{|\alpha|}{2\Lambda} + \frac{C^2}{4\Lambda^2}} \cos\left(2\sqrt{\Lambda}t + \delta\right) \tag{A.9}$$

which must have finite time-extent due to the zeros of $a(t)^2$.

A.2 Case $\alpha = 0$

For the given metric ansatz, the metric equations of motion are satisfied if and only if

$$\frac{\ddot{a}}{a} = -\frac{\Lambda}{2\lambda - 1} \tag{A.10}$$

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