Dynamical Systems, Symbolic Dynamics, and Measurement

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Abstract

This work offers a brief introduction to the topics of dynamical systems and symbolic dynamics. In particular, one dimensional maps of the unit interval are examined and some methods for visualizing partition development are discussed. Finally, possible applications of these ideas are discussed in the realm of measurement and instrumentation.

1 Dynamical Systems

1.1 Introduction

Perhaps one of the simplest and most important questions in the study of physics is that of how systems evolve in both time and space. How does a ball's position at some initial time affect its state at a later time? In what ways does the magnetic field surrounding a wire change as the wire's current is increased? Does a molecule's orientation correlate to the orientation of surrounding molecules? From Newton's laws of motion to Maxwell's equations for the electro-magnetic field, physical problems can largely be regarded as special cases encompassed in the broader mathematical area known as dynamical systems.

Loosely put, dynamical systems is the mathematical study of how things change. While problems in physics largely motivated the study of dynamics, the field is far reaching—extending into chemistry, biology, and even economics just to name a few. The general methods developed within the framework of dynamical systems are equally suited to describing changing populations of predators and prey within an ecosystem as they are in describing the motion of a block sliding down an inclined plane. What makes dynamical systems such a useful and general field is the idea that systems—physical, biological, etc.—can be distilled into a small number of mathematical components. Rather than considering positions, velocities, populations, and stock prices in a literal sense, we prefer to describing a system by an element of an abstract state-space along with a rule for how the system changes its state with time. In Newtonian mechanics the state-space consists of the set of all possible initial conditions (the phase space) while the rule for time evolution is given by a differential equation, namely the equation of motion. Alternately, for the logistic population model, the state-space is given by the unit interval $[0, 1]$ while the rule for how the population changes year to year is given by the simple difference equation:

$$x_{n+1} = rx_n(1 - x_n)$$

The power in this abstraction is that we no longer use some ad hoc mash of information to describe how a system behaves. Instead, we now characterize a system's evolution by a single trajectory in state-space from which all other system aspects may be derived. Roughly speaking, we may visualize a system as a point in some space tracing a curve as the system evolves with time. To further develop the above discussion, we now define these ideas within a more formal framework.

1.2 Vocabulary

Our first task in formalizing the ideas presented in the previous section is giving a concrete definition of a dynamical system.

A **dynamical system** is a triple $(\mathcal{M}, \tau, \mathcal{F})$ consisting of sets $\mathcal{M}$ and $\tau \in \{\mathbb{Z}, \mathbb{R}\}$ along with a function

$$\mathcal{F} : \mathcal{M} \times \tau \to \mathcal{M} \qquad (x, t) \mapsto \mathcal{F}(x, t)$$

with additional properties:

$$\mathcal{F}(x, 0) = x$$

$$\mathcal{F}(\mathcal{F}(x, t_o), t) = \mathcal{F}(x, t_o + t)$$

In fleshing out this definition, we first realize that the dynamical system is broken into three key components. The first of these sections, the set $\mathcal{M}$, is the state-space which we loosely mentioned before. Commonly we will take $\mathcal{M}$ to be some portion of the Euclidean space $\mathbb{R}^n$ (in the simplest case an interval of the real line) but in general we prefer the state-space be a manifold, hence the label. The second set in the triple consists of values which can be thought of as representing the times at which a system is defined. If $\tau = \mathbb{Z}$, we call the system a **discrete time dynamical system** while when $\tau = \mathbb{R}$ the system is referred to as a **continuous time dynamical system**. A model which predicts the size of an incoming freshman class based on the size of last year's class would be an example of a discrete time system as the system is defined only at discrete times, namely once a year. On the other hand, the position of a particle moving in a gravitational field would constitute a continuous time dynamical system as the particle's position and momentum are defined for every instant of time. As a final note, when considering the set τ one must understand that the elements of τ do not necessarily represent times. For instance, we may be analyzing the orientation of neighboring spins in an Ising spin chain and ask how an initial spin at position t_n is correlated to the spin at position t_{n+1}. In this case, time is irrelevant to the problem and the set τ is a set of positions, proving the point that care must be taken when considering the set τ. The final piece of a dynamical system is the function $\mathcal{F}$ called the **dynamic**. The dynamic can be thought of as a function which inputs some initial state x_o and outputs the new state of the system at time t. Often to emphasize that $\mathcal{F}$ is a function which evolves an initial state x_o to a time t, we will use the alternate notation:

$$\mathcal{F}^t(x) = \mathcal{F}(x, t)$$

As a final point on the definition, we consider the additional conditions placed on $\mathcal{F}$. The first merely states that a zero time represents no evolution of the system (i.e. that $t = 0$ corresponds to $x = x_o$). The second is more substantial in that it demands that the dynamic not be explicitly dependent on time—it may only depend on time differences. To be very specific, systems which satisfy this final condition are called **autonomous**.

Now that the definition for a dynamical systems has been given, we proceed by classifying some important points and subsets of the state-space $\mathcal{M}$. To begin, we give the definition of an orbit.

The **orbit** produced by a point $x \in \mathcal{M}$ is a subset $\mathcal{O}_x \subseteq \mathcal{M}$ defined as:

$$\mathcal{O}_x = \{\mathcal{F}^t(x) : t \in \tau\}$$

Informally an orbit may be thought of as the trajectory or curve that a point traces out as it moves along with time. Notice that orbits of autonomous systems do not intersect. This is due to the fact that at all times a point evolves to the next point in the orbit in the same way. If a point were to belong to multiple orbits it would mean there exists a point somewhere in the state-space with time dependent evolution.

An element $x \in \mathcal{M}$ is called a **stationary**, **periodic**, or **aperiodic point** respectively when:

$$\mathcal{F}^t(x) = x \qquad \text{for all } t \in \tau$$

$$\mathcal{F}^t(x) = \mathcal{F}^{t+T} \qquad \text{for some } T \in \tau$$

$$\mathcal{F}^t(x) \neq \mathcal{F}^{t+T} \qquad \text{for any } T \in \tau$$

An element $x \in \mathcal{M}$ is called a **wandering point** if there exists a neighborhood $\mathcal{N}_x \in \mathcal{M}$ and element $t_{min} \in \tau$ such that:

$$\mathcal{F}^t(x) \notin \mathcal{N}_x \qquad \text{for all } t \geqq t_{min}$$

Likewise an element $x \in \mathcal{M}$ which visits any open neighborhood containing itself infinitely many times is called a **recurrent** or **non-wandering point**.

It may seem that we now have all the language necessary to describe the evolution of a point in the state-space. Simply describe the system's trajectory and that is the end of the story. Unfortunately, the trajectory of a system is usually very hard to calculate and tells us little about the system's behavior. It is much more common to consider the evolution of groups of trajectories or equivalently volumes of the state-space. We do this because for many systems, individual trajectories are very sensitive to initial conditions. For systems of this type, the slightest error in measuring a system's initial condition leads to exponetially increasing error in calculating the system's current state. In fact after a characteristic time, trajectories which start arbitrarily close together diverge from one another and seem to have no correlation whatsoever. We call these types of systems chaotic. To better describe the evolution of bundles of trajectories we examine the idea of an attracting set.

A subset $\mathit{\Omega} \subseteq \mathcal{M}$ is said to be **attractive** if the image of $\mathit{\Omega}$ under the dynamic $\mathcal{F}$ is contained in $\mathit{\Omega}$. Alternately, this condition is given:

$$\mathcal{F}^t(\mathit{\Omega}) \subseteq \mathit{\Omega} \qquad \text{for all } t \in \tau$$

Attractors are important for determining the long term behavior of a dynamical system because trajectories which enter the attractor are forever trapped within. This means that while the dynamical system describes the evolution of points on the set $\mathcal{M}$, long term behavior is limited to a number of attracting subsets of the state-space. Before we become more bogged down in definition, we continue with some simple examples.

1.3 Example: One Dimensional Maps

In order to keep this work clean and accessible, we restrict ourselves to discrete time mappings of the unit interval called one dimension mappings. These systems are an ideal tool in the study of dynamical systems because the state-space is familiar and the dynamics can be visualized as any function contained within a unit box. Despite being extremely simple, one dimensional maps display many of the same interesting behaviors encountered in more complicated dynamical systems. To be specific, in this section we will look at four maps—the shift, tent, logistic, and parabolic shift maps.

1.3.1 The Shift Map

The shift map is a simple dynamical system $([0,1], \mathbb{Z}, \mathcal{F})$ where the dynamic is prescribed by:

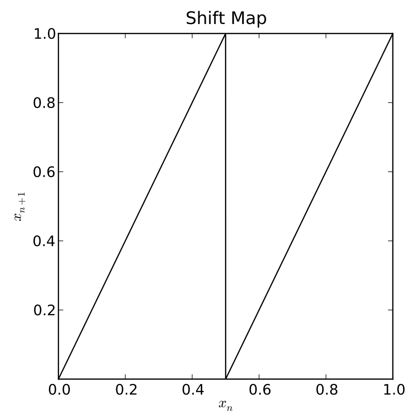
$$\mathcal{F}^1(x_o) = 2x \qquad \text{for all } x_o \in [0, .5]$$

$$\mathcal{F}^1(x_o) = 2x - 1 \qquad \text{for all } x_o \in (.5, 1]$$

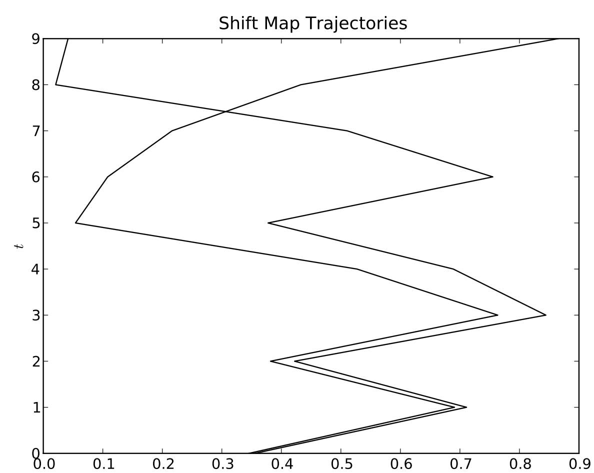
Note that we generally describe the dynamic using a rule for single step evolution. The dynamic as given in the definition of a dynamical system is obtained by compounding this rule for step evolution many times. Because of this point, we generally use the notation:

$$\mathcal{F}^1(x_n) = x_{n+1}$$

Using this notation we now show a plot of the shift map.



Despite seeming like a relatively safe function, the shift map exhibits chaotic behavior. We can get an intuitive feel for this by considering the evolution of two nearby points.



These two trajectories have an initial condition which differs by only .01 but by nine iterations of the map, the trajectories seem to have no semblance.

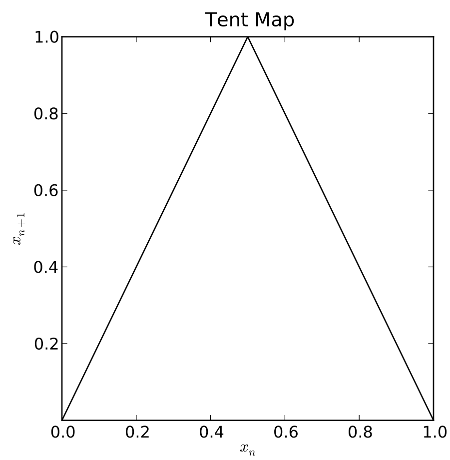
1.3.2 The Tent Map

Another one dimension map, the tent map, is described by the equations:

$$x_{n+1} = 2x \qquad\qquad \text{for all } x_o \in [0, .5]$$

$$x_{n+1} = 2(1 - x) \qquad\qquad \text{for all } x_o \in (.5, 1]$$

The map is defined in a similar way to the shift map with the exception that the function is inverted in the second half of the interval. We will later see that this small change has profound effects in the map's behavior.

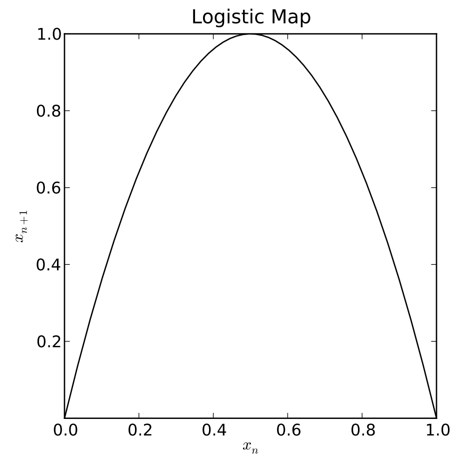


1.3.3 The Logistic Map

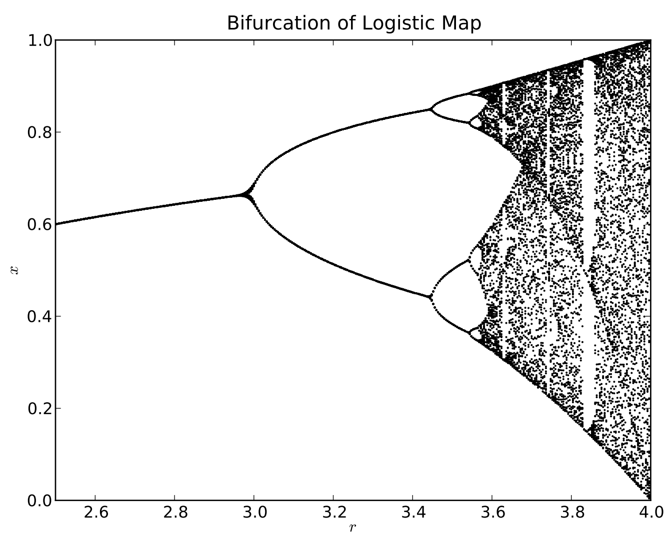
The logistic map is a one dimensional map which was used as an early model for population development. It is governed by the equations:

$$x_{n+1} = rx_n(1 - x_n) \qquad \text{for all } x_o \in [0, 1]$$

The value of r is chosen from the interval [0,4] because these values induce a function with the entire unit interval as its range. Below is a graph of the logistic map for the case of $r = 4$.



At this point it is a good idea to revisit the idea of an attractor. It turns out that the logistic map is not chaotic for all values of r. In fact for values of $r \in [0, 3.57]$ trajectories approach periodic orbits. At an r value of about 3.57, chaotic behavior sets in. To get an idea of how this transition occurs, we look at a bifurcation diagram for the logistic map.



In this figure we plot the r value on the x-axis and the attractor on the y-axis. Note how for small values of r the attractor is a single point meaning that most orbits approach a single point as the number of iterations tends to infinity. At an r value of about 3 we see that the attractor changes to a two point set. In this case the process approaches a period two process. This doubling continues forming longer and longer periods. By a value of $r = 4$ the period is infinite and the map is fully chaotic. Because of this doubling, we call this type of behavior the **period doubling route to chaos**.

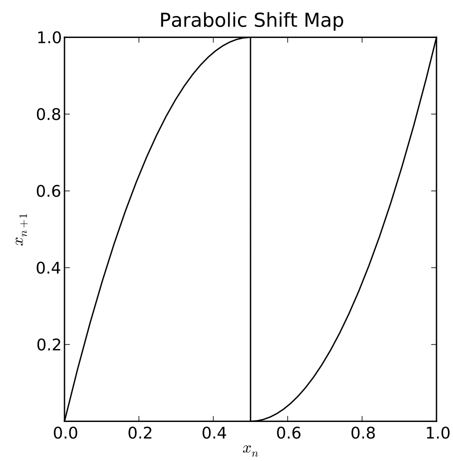
1.3.4 The Parabolic Shift Map

As a final example, we give a map which looks similar to both the shift and logistic map. As it was devised this summer, we call it the parabolic shift map. It is governed by the equations:

$$x_{n+1} = 4x_n(1 - x_n) \qquad \text{for all } x_o \in [0, .5]$$

$$x_{n+1} = (2x - 1)^2 \qquad \text{for all } x_o \in (.5, 1]$$

A plot of this function is given below.



1.3.5 Conjugation

With only a brief glance at the four maps presented previously in this section, one immediately begins to see some similarities. For instance, the function which describes the logistic map seems to be a deformation of the tent map. Likewise the shift and parabolic shift maps seem to be related to one another. This is no coincidence—these maps are related through a distortion of the interval induced by a bijective homeomorphism. Before continuing this discussion, we give the definition of conjugate maps.

Two one dimensional maps $\mathcal{F}$ and $\mathcal{G}$ are said to be **conjugate** if there exists a bijective homeomorphism h from the unit interval into itself such that:

$$h^{-1} \circ \mathcal{F} \circ h(x) = \mathcal{G}(x) \qquad \text{for all } x \in [0, 1]$$

Alternatively we may view this relationship in the form of a commutative diagram.

$$\begin{array}{ccc}
[0,1] & \xrightarrow{h} & [0,1] \\
\mathcal{G}\Big\uparrow & & \mathcal{F}\Big\uparrow \\
[0,1] & \xrightarrow{h} & [0,1]
\end{array}$$

In essence, when two maps are conjugate we can say that they are equivalent up to a transformation. When this transformation can be easily obtained, it is often easy to transform a map which is difficult to work with or not well understood into a map which is simple to manipulate. Then after computations are done, we can transform our solution back to the original state-space. Returning to the issue of the semblance between the tent and logistic maps, we have an example of conjugation. The conjugation function in this case is given by:

$$h(x) = \frac{1 - \cos \pi x}{2}$$

The corresponding commutative diagram is given as:

$$\begin{array}{ccc}
[0,1]_L & \xrightarrow{L} & [0,1]_L \\
h\Big\uparrow & & h\Big\uparrow \\
[0,1]_T & \xrightarrow{T} & [0,1]_T
\end{array}$$

Note that the subscripts T and L refer to the tent and logistic domains and functions. In a similar way we find that shift and parabolic shift maps are conjugate.

2 Symbolic Dynamics

2.1 Introduction

In this section we develop a very useful tool in the analysis of dynamical systems—symbolic dynamics. In some ways symbolic dynamics can be thought of as a bridge between measurements of a dynamical systems and the actual mathematical underpinning of the system. Symbolic dynamics attempts to answer how much information about a dynamical system can be drawn from a data sequence produced by measurements of the system. For example, when measuring the position of a harmonic oscillator with amplitude A, we do not literally measure $x(t)$ rather we make a number of discrete time, finite precision measurements. If our ruler has only one tick mark at its center and we assign the letter X to a measurement in the interval $[-A, 0)$ and the letter Y to the interval $[0, A]$, we might expect to obtain a sequence of measurements XXXYYYXXXYYYXXX.... when we make measurements at equal time intervals. Symbolic dynamics gives a method for converting real system trajectories into symbol sequences and then attempts to answer how much about the underlying system can be deduced from these sequences. In essence we divide up the state-space of the system, assign a symbol to each section of the state-space, and then create a symbol sequence by looking at how a system's orbit passes through these divisions. As we saw in the ruler case above, the measuring instrument induced a natural partitioning of the oscillators state-space. In the sections to come we will find there is in fact a direct correspondence between actual instruments and partitions of state-space. Finally, we will look at what we can learn about a system from a sequence of symbols and suggest some ways for assessing instrument quality.

2.2 A More Formal Development

To begin, we formalize the idea of dividing the state-space of a system by introducing the definition of a partition.

A **partition** is any set of open subsets $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots$ of the state space $\mathcal{M}$ with the property:

$$\bigcup_i \mathcal{M}_i = \mathcal{M}$$

$$\mathcal{M}_i \bigcap \mathcal{M}_j = \emptyset \qquad \forall\, i, j$$

Simply put, a partition is a set of non-overlapping patches of the state-space. In the case of one dimensional maps, all partitions are sets of intervals. Alternatively we can think of partitions as being defined by their boundary. In this way a partition of the unit interval would merely be a set of points making the divisions between partitions. Once a partition is in place, each subset is assigned a symbol from a set $\mathcal{A}$ called an **alphabet**. Now all the machinery is in place to assign each initial condition from $\mathcal{M}$ a symbol sequence. This can be done in a number of ways. In one method we consider the trajectory of a point $x \in \mathcal{M}$ at equal time steps. For each time step we merely append the symbol of the partition which the trajectory is currently in. Another method appends the symbol of a partition each time the trajectory enters a partition. Regardless of how we assign a sequence, the end result of the process is a mapping:

$$\Phi : \mathcal{M} \mapsto \mathcal{A}^{\mathbb{Z}}$$

One of the biggest advantages of using symbolic dynamics is the dynamics of a system become significantly simplified in the sequence space $\mathcal{A}^{\mathbb{Z}}$. In the sequence space the dynamic reduces to a mere shift in the sequence. For example:

$$\Phi(x_n) = ABCDBCCDJDECJCDE\ldots$$

$$\Rightarrow \Phi(x_{n+1}) = BCDBCCDJDECJCDE\ldots$$

We will refer to this shifting dynamic in the sequence space as the Δ function. The hope of symbolic dynamics is that one may convert a dynamical system into a symbolic system, study the simplified dynamics in sequence space, and then take the results back to the state-space of the original system. In other words we hope to find a way of forming our sequences such that Φ^{-1} exists. Put in terms of a communative diagram:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\mathcal{F}"] \arrow[d, "\Phi"'] & \mathcal{M} \\
\mathcal{A}^{\mathbb{Z}} \arrow[r, "\Delta"] & \mathcal{A}^{\mathbb{Z}} \arrow[u, "\Phi^{-1}?"]
\end{tikzcd}$$

In the next section, it will be discussed when Φ is a "good" mapping which retains all the relevant information about the original system.

2.3 Instuments and "Good" Partitions

In addition to simplifying the dynamics of a given system, symbolic dynamics also is important in that it gives a relationship between dynamical systems and measurement. In general, any instrument which we use to measure a system spits out a sequence of finite precision numbers. These numbers can then be interpreted as a symbolic trajectory of the original dynamical system. In this way, we conclude that every instrument measuring a dynamical system induces a natural partitioning of the system's state-space. By viewing the partition induced by an instument, we may begin to assess an instrument's ability to extract information about the original system. Most notably we ask whether a systems initial state can be concluded from the symbolic sequence produced by the instrument. This comes down to a question of the existence of Φ^{-1}. Take for instance an instrument which trivially partitions the state space into a single component. In this case Φ^{-1} clearly does not exist as any system trajectory will produce the constant symbol sequence AAAAAAAA...... (assuming A is the symbol assigned to the partition and measurements are made at equal time intervals). So we know in general that the transition from a system's true dynamics to a symbolic representation is not invertible. In fact for most instruments, this transition is not invertible.

Generally there are two ways to produce instruments which come closer to telling us a system's true internal state. The first and perhaps more obvious procedure is to use a finer partitioning. This scenario has the advantage that we learn a lot about the system using only a few measurements. In fact, in the limit that the state-space is infinitely partitioned, a measurement would tell us the exact system state with a single measurement as each partition would consist of an infinitesimal neighborhood surrounding a possible state. The downside to using such an instrument is that one must store and manipulate an alphabet with a huge amount of symbols. Also if one needs only a rough approximation to the system's internal state, a large amount of measurement power is wasted.

A second approach to this problem is the idea of using a so called generating partition. In this scenario rather than using a very fine partition, we use a partition for which we can learn more and more with each additional measurement. Then in the limit of an infinite number of measurements, we will learn the exact system state. More formally, a generating partition is one in which Φ^{-1} exists (Φ is injective) and we lose no information when making the transition from system trajectories to symbolic trajectories. This scheme has the advantage that one can make measurements until a system's state is known to a given tolerance. Additionally a generating partition often consists of only a few partitions so, in general, fewer symbols need to be stored. As a final note, it is pointed out that these two strategies may be employed simultaneously when designing a measurement device. Because a further partitioning of a generating partition is still generating, and one may start with a generating partition and then make further subdivisions.

2.4 Visualizing Partitions: Encoding Symbol Sequences

The bulk of the research for this project was aimed at finding an effective way to visualize the development of an instrument's natural partition. In other words, the work focused on visualizing exactly how subsequent measurements allow a viewer to increasingly discern the different sets of initial system configurations. For instance if the system were an oscillator, this would be equivalent to asking after N measurements to what area of the phase space can we deduce the system was initially in. Note that in general this area will depend on the both the system's initial position and the measuring device. Hence we want a way of looking at the development of the entire state-space rather than specific system trajectories.

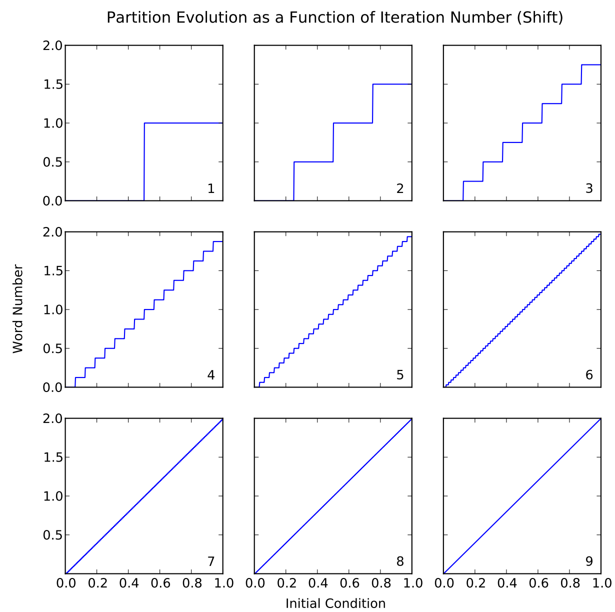
For this work, the main focus was on finding a way to visualize this development for one dimensional maps as these dynamical systems are simple to study and the low dimensionality of their state-space lends itself well to visualization. The method chosen to accomplish this task was a numerical encoding of system trajectories. Perhaps the best way to describe this process is with an example. Take for instance the logistic map with the partition $\{[0, .5), [.5, 1]\}$. We will assign the symbols 0 and 1 to the first and second partitions respectively. Now in general, an initial state of the system will trace out some symbolic trajectory such as 10110100000000...... We choose a simple coding scheme to display these symbolic trajectories. In essence, the symbolic trajectory is sent to a number in the interval $[0, 2]$ by treating the trajectory as a base 2 representation of a number. As an example, consider the trajectory 10110100000000......

$$
\begin{aligned}
1011010000000 \qquad \longrightarrow \qquad & 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} + 0 \times 2^{-4} \\
& = 1.40625
\end{aligned}
$$

In this way every symbolic trajectory is mapped to a real number. By looking at subsequences (i.e. the first N symbols of each trajectory), it is possible to see which system states an observer is able to differentiate between at each time step. Finally, by looking at these mappings it is also possible to qualitatively judge just how "generating" a partition is as these partitions will approach injective functions in the infinite symbol limit.

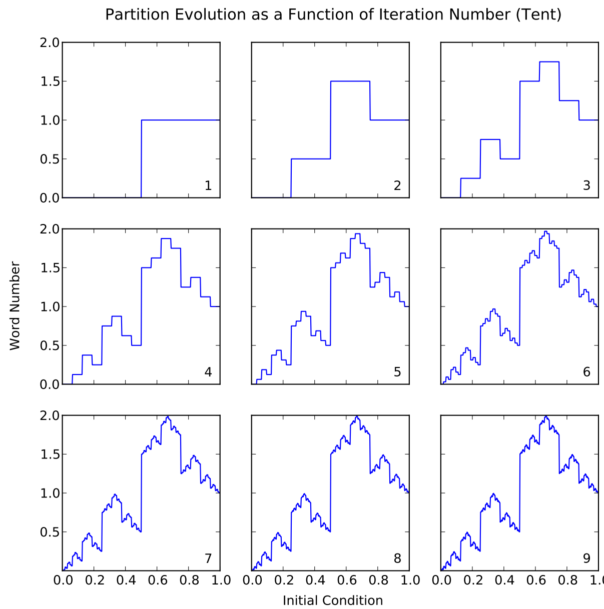
2.5 Results: Visualizing Partition Development for 1-D Maps

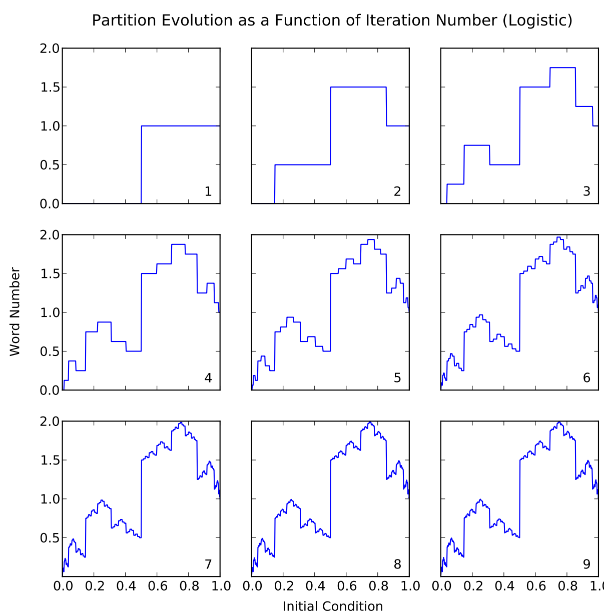
In this section, the development of partitions is examined for the shift, tent, logistic, and parabolic shift maps. To begin we look at this development of the shift map when the initial partition is taken to be $\{[0, .5), [.5, 1]\}$. This partition is chosen as it has the generating property.



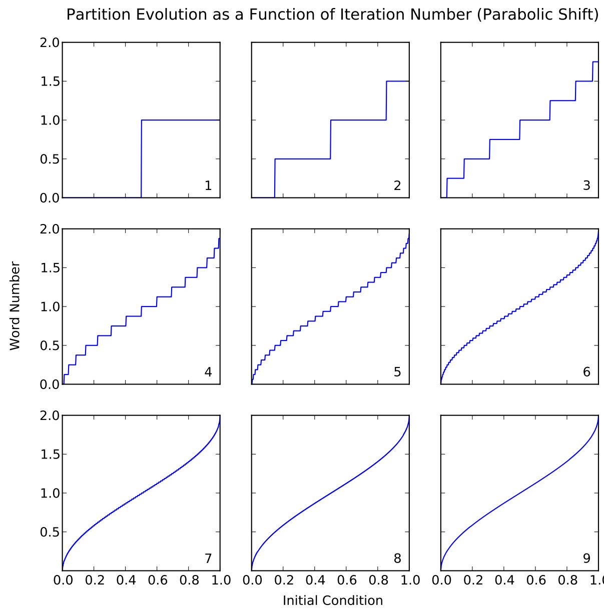
In this plot, each panel gives the numerical encoding of the initial conditions after a certain number of iterations. As an example, consider that for two iterations the possible symbol sequences are 00, 01, 10, and 11. This is indicated in the second panel as there are four distinct steps corresponding to the four possible two symbol sequences. Note that as additional iterations (measurements) are made on the system, more and more steps occur and hence one can restrict the set of possible system states to a smaller and smaller interval. Looking at the encoding after just 9 iterations it becomes apparent that in the limit of an infinite number of iterations, the function becomes a straight line bijection. We conclude the initial partition is in fact generating as stated earlier. The following are plots of the tent, logistic, and parabolic shift maps for the same partition.

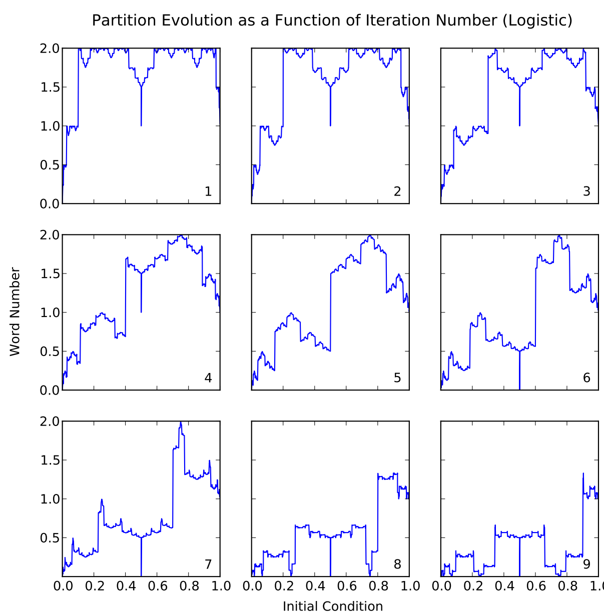
Note that for both the tent and logistic maps the partition becomes fractal in nature. Despite this aspect the encoding is bijective (although extremely discontinuous). Also when looking at the partition evolution for the tent and logistic maps or the two shift maps, note the similarity in the functional shape.





The evolution of the logistic map seems to be a distortion of the tent map. This





occurs as these maps are conjugate to one another. It might be possible to use visualizations like these to pick out maps that are related by such transformations. Finally, we look at what happens to the evolution of the logistic map as the partition is moved away from .5 (and hence is no longer generating).

In this figure, the number at the bottom of each panel corresponds to the partition placement. For instance the number 8 corresponds to a partition at .8. Notice how quickly the devolvement degrades as the partition is moved away from .5 and hence is no longer generating. In these degraded cases, large flat plateaus form indicating that there are a large numbers of initial conditions which cannot be differentiated. Also some intervals become disconnected. In some cases the interval of possible initial conditions is very small but the interval is highly disconnected and spread over a large region. Measurements of this type would be of little use and can be detected using this type of visualization method.

2.6 Conclusions and Possible Uses

In this work we examined the transition of a dynamical system to a discrete symbolic system. The connection between this discretization and measurement instruments was laid out in the language of symbolic dynamics. The main focus of this summer's research was developing effective methods for visualizing how the natural partition of an instrument leads to increased knowledge of the underlying system.

In the future it would be interesting to employ such techniques on simple (possibly even just theoretical) instruments which measure real physical systems. One example might be a bead in a tube which is attracted to one side or another based on surrounding forces. Looking at how these theoretical instruments fair may give insight into the design of real instruments in the future. Also the approach of using generating partitions in instrumentation may someday be applied to making instruments which adaptively change to maximize the amount of information content per measurement. Such an instrument would adapt to bring its natural partition closer to that of a generating partition.

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