Simulation of Boundaries in (2+1)-Dimensional Causal Dynamical Triangulations

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Abstract

In this paper, we will present some algorithms for numerically approximating $(2+1)$-dimensional quantum spacetimes with fixed spatial boundaries, according to the theory of Causal Dynamical Triangulations. The methods we will describe allow for integration over the causal spacetimes that interpolate between two arbitrary, fixed spatial configurations of known temporal separation.

1 Introduction

1.1 The Path Integral Approach to Quantum Gravity

There is no accepted quantization of gravity. The most widely-accepted theory of gravity is General Relativity, which is classical. One way to develop a quantum theory from a classical theory is to exploit an action principle of the classical theory to devise a path integral [5]. In such a quantization, one computes the amplitudes for transitions between configurations by adding up terms of the form e^{iS}, where S is the classical action of some path between configurations. One such term is added for each possible path. The set of paths is usually an uncountable continuum, so the summation becomes an integral over all possible paths.

The simplest example of a path integral formulation is the quantization of single-particle mechanics, in which the configurations are positions, x_a and x_b, at times t_a and t_b. The set of paths between these configurations is all valid spacetime trajectories starting at a and ending at b. The action, S, is just the time-integral of the Lagrangian. Integrating e^{iS} over all paths, then normalizing, gives the probability amplitude of observing a particle at point b, given that it was observed at point a.

Just as Newton's Laws result from extremizing the actions of particle trajectories, the Einstein Equations of General Relativity result from extremizing the Einstein-Hilbert action, S_{EH}. In vacuum, this action is the integral, over all spacetime, of the difference between the curvature scalar and twice the cosmological constant:

$$S_{EH} = \int d^n x \sqrt{-g}\ (R - 2\Lambda)\ . \tag{1}$$

In general, S_{EH} is a functional of the spacetime metric, g. This suggests that we might calculate amplitudes in a theory of quantum gravity using path integrals of the form

$$N \int \mathscr{D}g\ e^{iS_{EH}}$$

where $\int \mathscr{D}g$ is shorthand for integration over some set of spacetime metrics and N is a normalization factor. The primary theoretical challenge in developing such a quantization is determining exactly what set of spacetime metrics to integrate over. In this paper, we will discuss integrations over sets of spacetimes that interpolate between space-like boundaries. We assume that such integrals give transition amplitudes between spaces [5]. Further, we restrict the region of integration to causal spacetimes, as dictated by the theory of Causal Dynamical Triangulations.

1.2 Causal Dynamical Triangulations

The theory of Causal Dynamical Triangulations (CDT), developed by Ambjørn, Loll, and Jurkiewicz [3], restricts possible spacetimes to those which enforce causality. This means that permissible spacetimes must allow a global time function and foliation into spatial surfaces that don't change topology over time.

Most studies on CDT have used discrete approximations with integral numbers of space-like slices, each representing a hypersurface of constant time. These studies approximate the spacetime manifold as a simplicial manifold. Simplicial manifolds generalize triangular tessellations of 2D surfaces to higher dimensions; curved 3D manifolds may be "triangulated" into tetrahedral simplices, and so on. Simplicial manifolds of dimension d have all curvature focused on their $(d-2)$-dimensional "bones" (points in the 2D case, edges in the 3D case, and so on). The curvature tensor is therefore only nonzero at the bones, reducing the integral over spacetime for S_{EH} to a summation over the bones [4]. We can simplify the form of the action further by holding edge lengths (and thus bone areas) constant and equal. With that assumption, the action becomes just a linear function of the numbers of bones and d-simplices [1].

By introducing a global time coordinate, CDT provides a clearer prescription for Wick rotation than other, similar theories with less-restricted sets of possible paths. We can replace the global time coordinate with imaginary time (the substitution $t \to it$), as in more familiar Wick rotations. In CDT, rotation to imaginary time amounts to using imaginary timelike edge lengths. This substitution is extremely useful for Monte Carlo simulations of CDT, because it makes the action purely imaginary [1]. Multiplying this imaginary action by

i in the exponential terms contributed by individual histories in a path integral produces an integral of the form

$$\int \mathscr{D}g \; e^{-S} \; . \tag{2}$$

The above is a statistical partition function, giving real-valued weights for different paths. We can use these weights to come up with probabilities for accepting or rejecting randomly-generated geometries in a Monte Carlo approximation of CDT's path integral. The quantity S in the Wick-rotated partition function is, conveniently, the Einstein-Hilbert action of a manifold with positive definite metric [2]. Moving the problem into a Euclidean space allows us to dodge the complications of treating time- and space-like distances and angles separately.

1.3 Boundary Terms in the Action

I suggested earlier that the Einstein-Hilbert action (in vacuum) was just (1) but that formula is true only on manifolds without boundaries; its derivation throws away a term in the variation of the action that vanishes absent boundaries (cf. Wald [6], Appendix E). Wald goes on to discuss the action of a manifold with fixed boundary geometry. In that case, we must add an additional term to the action to recover the Einstein equations after extremization. The extra term is twice the integral of extrinsic curvature over the boundary:

$$S_{EH} = \int\limits_{M} d^n x \sqrt{-g} \; (R - 2\Lambda) \; + \; 2 \int\limits_{\partial M} d^{(n-1)} x \sqrt{\pm h} \; K \; .$$

In the above expression, M represents the spacetime manifold, ∂M denotes the boundary, h is the induced metric on ∂M, and K is its extrinsic curvature. The signature of h depends on whether ∂M is space-like or time-like, so its determinant may require negation to ensure a real-valued volume form.

The action of a simplicial spacetime must also change to accommodate boundaries. Hartle and Sorkin [4] discuss the additional term needed for the action of a simplicial manifold of positive definite metric with fixed boundaries. As the action for closed simplicial manifolds turned out to be a sum over interior bones, the additional boundary term is just a sum over boundary bones.

2 Methods

2.1 Overview of Numerical Simulation

We approximated the CDT path integral by randomly sampling Wick-rotated Euclidean geometries, according to the weights given by (2). We implemented this technique in a Common Lisp program, storing geometric primitives (triangles and tetrahedra) as lists of adjacency information. The program associates each primitive with a unique integer key through a hash table, providing fast lookup, addition, and removal of geometry. The program samples the space of triangulations by randomly selecting small, local changes to the geometry, drawn from a pool of several ergodic "moves". The program decides whether to apply a given move by way of a Metropolis algorithm: it computes the move's effect on the action of the simplicial manifold and randomly decides whether to accept or reject the move in such a way as to enforce (2).

2.2 Simulation of Boundaries

I mentioned in 1.3 that boundaries add a term to the Einstein-Hilbert action. Further, I noted that the simplicial approximation of this term takes the form of a summation over boundary bones. In general, this term is [4]

$$\sum_{b\in\partial M} V(b)\left(\pi - \sum_{\sigma\supset b}\Theta(\sigma;b)\right) ,$$

where $b \in \partial M$ is a bone in the boundary of the simplicial manifold, $V(b)$ is its volume (length in the $(2+1)$ case), $\sigma \supset b$ is a simplex containing bone b, and $\Theta(\sigma;b)$ is the dihedral angle about b occupied by σ.

In our simulation, we assume that all simplices are equilateral tetrahedra of unit edge length, so $V(b)$ is unity and Θ has the constant value of $\arccos(1/3)$. There are two ways that a simplex can have one or more bone on the boundary: it can have a single edge or an entire face on the boundary. Each simplex with a single edge on the boundary will contribute $\arccos(1/3)$ to the sum over dihedral angles and each simplex with a face (and thus three edges) on the boundary will contribute $3\arccos(1/3)$. In our simulation, then, the boundary term of the action becomes

$$N_b\pi - (N_1 + 3N_3)\arccos\left(\frac{1}{3}\right) .$$

where N_b is the number of boundary edges, N_1 is the number of tetrahedra with a single edge on the boundary and N_3 is the number of tetrahedra with three edges on the boundary. To simulate the effect of this term, we must keep track of changes to the numbers N_b, N_1, and N_3. For fixed boundary geometries, only N_1 may actually change, making the boundary term a linear function of one variable.

In our program, we associate each simplex with the time stamps of its earlier and later spatial slices, allowing us to detect when we perform a move on geometry next to a boundary. The program then updates N_1 appropriately.

2.3 Initialization of Boundaries

The Monte Carlo algorithm described in 2.1 is a random walk through our region of integration. Generating a starting point for that random walk is a non-trivial problem. Here, we will discuss generation of 2+1 spacetimes with spherical (S^2) spatial topology. In the case of periodic boundary conditions, we began with a single tetrahedra at each time value and used a manually-derived simplicial

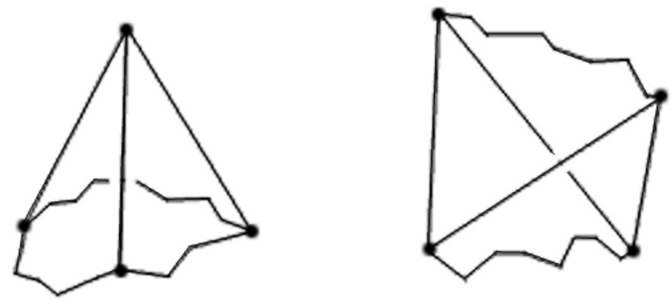

Figure 1: Possible types of complexes. The jagged lines are space-like pseudo-edges. The straight lines are time-like edges.

filling between consecutive tetrahedral spaces. We cannot work out by hand, though, the fillings between arbitrary S^2 triangulations required for general-purpose simulation of manifolds with fixed S^2 boundaries. Below, we discuss an algorithm for generating sets of tetrahedra that form simplicial manifolds between two arbitrary S^2 boundaries.

Our approach is to generalize our known triangulation between two tetrahedra. We do this by breaking each S^2 sheet into four simply-connected regions we call "pseudo-faces", each bounded by a ring of line segments arbitrarily split into three contiguous "pseudo-edges". These pseudo-edges are analogous to the sides of triangular facets of tetrahedra. The pseudo-faces meet each other at pseudo-edges with the same combinatorics as the faces of a tetrahedron meeting at edges. We replace the simplices used to interpolate between tetrahedra with analogous complexes in which space-like triangles are replaced by pseudo-faces, space-like edges are replaced by pseudo-edges, and time-like triangles are replaced by the triangle fans formed by connecting all points of a pseudo-edge to a vertex at which two pseudo-edges meet on the other spatial slice. There are two types of complexes formed in this decomposition (see Figure 1): ones with a pseudo-face on one sheet and a vertex on the other and ones with a pseudo-edge on each sheet. Each type of complex has a straightforward and unique decomposition into simplices.

A challenge that we glazed over above is how to decompose a triangulation of S^2 topology into pseudo-faces and pseudo-edges. The simplest algorithm to do this is to select a random triangle as the first pseudo-face, select a neighbor of that triangle as the second pseudo-face, specify all other triangles meeting at one endpoint of their shared edge as the third pseudo-face, then designate all remaining triangles as the fourth pseudo-face (see Figure 2). Making the five edges of the first two triangles pseudo-edges uniquely determines the sixth pseudo-edge.

The idea of decomposing spatial sheets into pseudo-faces and pseudo-edges

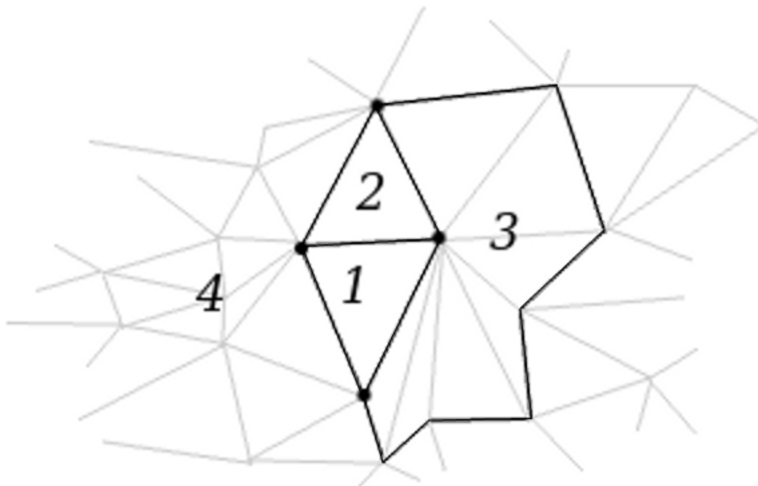

Figure 2: Simple decomposition of an S^2 triangulation into pseudo-faces and pseudo-edges.

may be extended to other topologies, such as the 2-torus, but we have not yet formulated a precise algorithm for geometries other than the 2-sphere.

3 Results

3.1 Preliminary Testing

To test the stability of the program, we first ran a number of small-volume (around 8,000 simplices) test cases. These allowed us to detect any major bugs and verify that the program was generating consistent geometries. We then ran a larger volume test with 80,000 simplices in which each of the spatial boundaries was set to a single tetrahedra (the minimal triangulation of a 2-sphere). Because we are specifying the same geometry for the first and last slice, we are technically integrating over a subset of the geometries allowed by specifying periodic boundary conditions (in which the last and first time slices are identified with one another). In practice, periodic boundary conditions give rise to integrals dominated by sphere-like geometries (corresponding to the Wick-rotated de Sitter space), each with a thin handle (forced by the $S^2 \times S^1$ topology) [2]. With the boundary conditions in our test case, we are essentially forcing one particular slice to be minimally-small. We expect, then, to get an integral dominated by geometries that are largely spherical, but have thin tubes (corresponding to the thin handle of the periodic case) extending to either boundary.

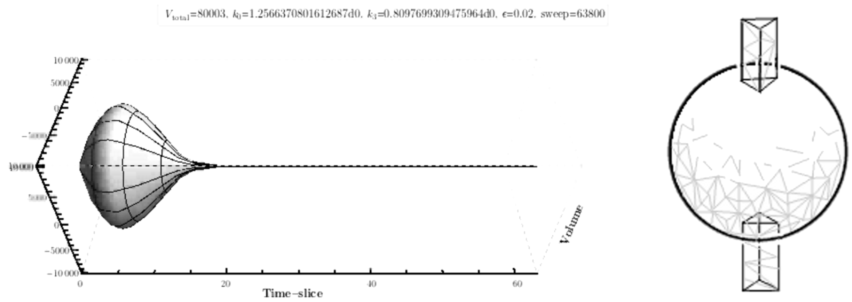

Figure 3: To the left is an example of a sphere-like spacetime, generated in the random walk through geometries. Compare with a 2D schematic representation of our expected result, as shown on the right.

We have not yet rigorously analyzed the results of this large-volume test, but they have the qualitative features we expected. Figure 3 illustrates one of the spacetimes from the integral and gives an example of the dominant type of geometry in the random walk.

Note that the visualization of spacetime in Figure 3 shows only the volume distribution per time slice and should not be interpreted as a literal depiction of the geometry. Recall that each spatial slice is actually a closed manifold of topology S^2. This permits no straightforward representation in flat, 3D, Euclidean space.

Generating large boundary geometries of physical significance is a challenge outside the scope of this paper. We have tested some non-trivial boundary geometries for technical purposes, but these have been highly artificial spatial configurations, generated without any physical interpretation in mind.

4 Conclusion

We have devised and implemented a method for simulating the theory of CDT in $(2+1)$ dimensions with fixed space-like boundaries. We have tested this method on some simple test cases, as discussed in 3.1. Further work might revolve around deciding what fixed boundaries to use as inputs to the program and extending our algorithms to other topologies and dimensionalities.

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