Practice Midterm Answers
Physics 104A

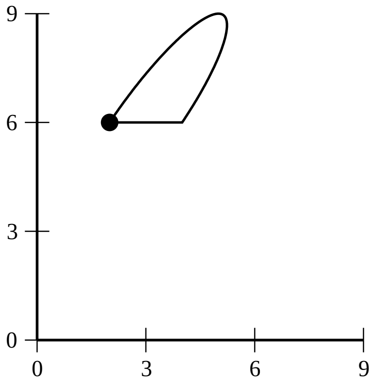
1. The argument of the δ-function (I'll call it $g(x)$) vanishes at only two points on [0,2]: $x = 1/2$ and $x = \pi/2$. Evaluate $\frac{x^2-1}{|g'(x)|}$ at each of these x-values and add the two contributions to get $\frac{-3}{8\cos 1/2} + \frac{\pi^2-4}{4(\pi-1)}$.

2. a) Most points aren't on the plane. Some safe ones to pick are those in the $y = 1$ plane that are not also in the $x = z$ plane, for example (0,1,1).

b) Any differences between the given three points are in the plane, for example (1,1,1)-(0,1,0)=(1,0,1) and (1,2,3)-(1,1,1)=(0,1,2). These two aren't orthogonal though. You can make an orthogonal basis for the plane through Gram-Schmidt: normalizing (1,0,1) gives $\frac{1}{\sqrt{2}}(1,0,1)$; then subtract the component of (0,1,2) in this direction to get $(0,1,2) - \sqrt{2}\frac{1}{\sqrt{2}}(1,0,1) = (-1,1,1)$. So (1,0,1) and (-1,1,1) are orthogonal in-plane vectors.

c) Pick a vector not in the plane, e.g. (1,1,1)-(0,1,1)=(1,0,0). Next subtract off the contributions along the two in-plane vectors, $(1,0,0) - \frac{1}{2}(1,0,1) + \frac{1}{3}(-1,1,1) = (\frac{1}{6}, \frac{1}{3}, -\frac{1}{6})$. This vector (or (1,2,-1), which is in the same direction) is orthogonal to the plane. You can check quickly that it's indeed perpendicular to the original two in-plane vectors from part b). (In fact, you could really have come up with the answer without the Gram-Schmidt, just by playing a bit with the vectors involved, since they happen to be so simple.)

3.

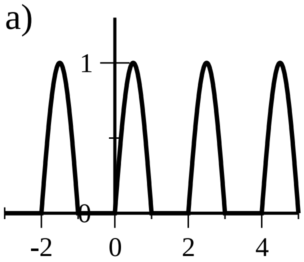


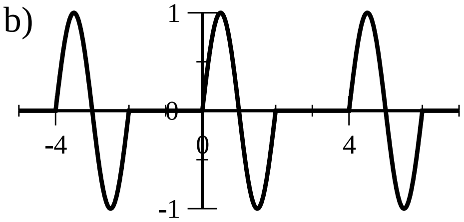
4. a) Multiply the original expansion by itself, term-by-term, to get $1 + 2x^2 + 4x^4 + 8x^6 + 16x^8 + \ldots$.

b) The series found in a) is a geometric series, $1 + y + y^2 + y^3 + \ldots$ with $y = 2x^2$, so when it converges it equals $\frac{1}{1-2x^2}$. This is the square of the original series, so the original series converges to $\frac{1}{\sqrt{1-2x^2}}$.

c) The sum function has a singularity at $x = 1/\sqrt{2}$, so it can't converge when $|x| > 1/\sqrt{2}$, or $|x|^2 > 0.5$. At the point in question $|x|^2 = \frac{4}{9} + \frac{1}{4} = \frac{25}{36} > \frac{1}{2}$, so the series does not converge.

5. a) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$, with $c_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}dx = \frac{1}{4i}\int_0^1 (e^{i\pi x} - e^{-i\pi x})e^{-in\pi x}dx = \frac{1}{4i}\int_0^1 (e^{i(-n+1)\pi x} - e^{i(-n-1)\pi x})dx$. For $n = 1$, this works out to $1/4i$; for $n = -1$, it becomes $-1/4i$. For other odd n, $c_n = 0$, while for even n it is $-1/\pi(n^2 - 1)$. (To get that you need to use $e^{i\pi} = -1$. Other than that it's standard integration and algebra, although it was still the messiest problem on this exam.)





b) $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi}{2}nx$, with $b_n = \int_{-1}^{1} f(x)\sin\frac{\pi}{2}nx dx = \int_0^1 \sin\pi x \sin\frac{\pi}{2}nx dx$

c) Just find anything orthogonal to f on [0,1] and continue it symmetrically to [-1,0]; for example, the function $\sin 2\pi|x|$ works.

6. $(ZC - CZ)f = zf^* - (zf)^* = zf^* - z^*f^* = (z - z^*)f^* = (2i\text{Im}z)f^* = [(2i\text{Im}z)^*f]^* = [(-2i\text{Im}z)f]* = CMf$, where M is multiplication by $-2i\text{Im}z$.

7. a) If A has a set of orthogonal eigenvectors, you can always normalize them and use them to write down a unitary matrix S. (If A is real, S will be orthogonal.)

b) If the vectors that form the columns of S aren't normalized, then S won't be unitary. A more interesting case is when A happens to have degenerate eigenvalues. Then any linear combination of the corresponding eigenvectors is also an eigenvector, so there will exist non-orthogonal sets of eigenvectors. Using such a set would give a non-unitary S.

c) $(S^{-1}AS)^\dagger = (S^\dagger AS)^\dagger = S^\dagger A^\dagger S$. (First equality because S is unitary, second because Hermitian conjugate of a product is the product of the Hermitian conjugates in reverse order.) Also, $(S^\dagger AS)^\dagger = (S^\dagger AS)^* = S^\dagger AS$, where the first equality is because $S^\dagger AS$ is diagonal and the second because the eigenvalues of A (i.e., the diagonal entries of $S^\dagger AS$) are real. From $S^\dagger A^\dagger S = S^\dagger AS$, multiply on the left by S and on the right by $S^\dagger$ to get $A^\dagger = A$, or A is Hermitian.

d) The more general condition is simply $S^\dagger A^\dagger S = (S^\dagger AS)^* = S^{*\dagger}A^*S^* = S^TA^*S^*$.