Midterm Answers
Physics 104A, Fall 2009

1. a) This is a geometric series with the ratio of term $n+1$ to term n equal to $\frac{x-2}{3}$. As long as this quantity has magnitude less than one, the power series converges. For complex x, that means $|x-2| < 3$, or x lies in a circle of radius 3 centered at 2. Along the real axis, this means $-1 < x < 5$.

b) The circle described in part a) intersects the imaginary axis at $\pm b$, where $b = \sqrt{3^2 - 2^2} = \sqrt{5}$, so the interval is from $-\sqrt{5}i$ to $\sqrt{5}i$.

c) Use the ratio test: $\frac{\sqrt{(2(n+1))!}/(n+1)!}{\sqrt{(2n)!}/n!} = \sqrt{(2n+2)(2n+1)}/(n+1)$. As n grows large, this approaches $\sqrt{(2n)(2n)}/n = 2$, so the terms of the series are growing and it can't possibly converge.

2. The argument of the delta-function factors to $x(3x-1)(x+1)$, which vanishes at $x = 0$, $x = 1/3$, and $x = -1$. The derivative of the argument is $9x^2 + 4x - 1$, which evaluates to -1, 4/3, and 4, respectively, at the three zeroes. Now evaluate the rest of the integrand at each of the zeroes of the delta-function argument: 1, $\cos\pi/6 = \sqrt{3}/2$, and 0. Combine to get: $\int_{-\infty}^{\infty} \delta(3x^3 + 2x^2 - x)\cos\frac{\pi}{2}x dx = \frac{1}{|-1|} + \frac{\sqrt{3}/2}{|4/3|} + \frac{0}{|4|} = 1 + \frac{3\sqrt{3}}{8}$.

3. By the definition of orthogonality, $0 = \int_0^L (x^2 + x + 1)(x-1)dx = \int_0^L (x^3 - 1)dx = \frac{1}{4}x^4 - x|_0^L = \frac{1}{4}L^4 - L$. Solving for L gives $L = 4^{1/3}$.

4. a) Need sines and cosines that are periodic with period $\sqrt{2}$, which means $\sin\sqrt{2}\pi nx$ and $\cos\sqrt{2}\pi nx$.

b) $f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n \cos\sqrt{2}\pi nx + b_n \sin\sqrt{2}\pi nx)$

c) $b_5 = \sqrt{2}\int_0^{\sqrt{2}} \sin 5\sqrt{2}\pi x f(x)dx = \sqrt{2}\int_0^1 \sin 5\sqrt{2}\pi x e^x dx + \sqrt{2}\int_1^{\sqrt{2}} e \sin 5\sqrt{2}\pi x dx$. The second term is $\sqrt{2}e(-\frac{\cos 5\sqrt{2}\pi x}{5\sqrt{2}\pi})|_1^{\sqrt{2}} = \frac{e}{5\pi}(\cos 5\sqrt{2}\pi - 1)$. For the first term, call the integral I and integrate by parts twice. This will give some boundary terms, plus the original integral: $I = \int_0^1 \sin 5\sqrt{2}\pi x e^x dx = -\frac{\cos 5\sqrt{2}\pi x}{5\sqrt{2}\pi}e^x|_0^1 + \int_0^1 \frac{\cos 5\sqrt{2}\pi x}{5\sqrt{2}\pi}e^x dx = \frac{1}{5\sqrt{2}\pi}(1 - e\cos 5\sqrt{2}\pi) + \frac{\sin 5\sqrt{2}\pi x}{50\pi^2}e^x|_0^1 - \int_0^1 \frac{\sin 5\sqrt{2}\pi x}{50\pi^2}e^x dx = \frac{1}{5\sqrt{2}\pi}(1 - e\cos 5\sqrt{2}\pi) + \frac{e \sin 5\sqrt{2}\pi}{50\pi^2} - \frac{1}{50\pi^2}I$. Solving for I, $I = \frac{5\sqrt{2}\pi(1-e\cos 5\sqrt{2}\pi)+e\sin 5\sqrt{2}\pi}{50\pi^2+1}$. Plugging this back into the equation for b_n gives $b_n = \frac{10\pi(1-e\cos 5\sqrt{2}\pi)+e\sqrt{2}\sin 5\sqrt{2}\pi}{50\pi^2+1} + \frac{e}{5\pi}(\cos 5\sqrt{2}\pi - 1)$.

5. a) Just multiply each eigenvector by the first row of the matrix to find the first entry of the product. (Assuming you believe that they really are eigenvectors.) For example, -9+64-44+4=15, which is 15 times the corresponding entry in the eigenvector, so the eigenvalue is 15. The other three eigenvectors give (in order) 27, 27, and 15.

b) Because there are two pairs of degenerate eigenvectors, the most general eigenvector is $A(1,2,1,1) + B(1,-2,-2,0)$ or $C(-1,0,1,2) + D(1,1,0,1)$, where A, B, C, D are any constants. Here I have resorted to writing row vectors rather than column vectors to save space.

c) There is one orthogonal eigenvector in the same plane of eigenvectors. In addition, there is one orthogonal eigenvector in the other plane. In principle (1,-2,-2,0) could have been orthogonal to the entire other plane of eigenvectors. This would *have* to be the case for a real symmetric matrix, which the beast I gave you clearly isn't. But in fact (1,-2,-2,0) is not perpendicular to the other plane, since for example it isn't orthogonal to (1,1,0,1).

d) For the same-plane eigenvector, do one round of Gram-Schmidt: start with (1,2,1,1), and subtract off the component along (1,-2,-2,0). This will give a vector in the plane, which will also be perpendicular to (1,-2,-2,0). The calculation is: $(1,2,1,1)-(-5)\frac{1}{9}(1,-2,-2,0) = \frac{1}{9}(14,8,-1,9)$

For the eigenvector in the other plane, you could engulf yourself in a truly horrendous calculation by finding a pair of orthonormal vectors in the plane, projecting (1,-2,-2,0) by taking the sum of its components along the orthonormal in-plane directions, and finally by constructing another in-plane vector orthogonal to the projection. Yucko. Instead, since the eigenvectors I gave are fairly simple, note that you're looking for a vector $a(-1,0,1,2)+b(1,1,0,1) = (b-a,b,a,2a+b)$ that is orthogonal to (1,-2,-2,0); that is, $0 = (b-a)-2b-2a+0 = -b-3a$, or $b=-3a$. I'll take $a=1$, which gives the vector $(-4,-3,1,-1)$. As a linear combination of eigenvectors with eigenvalue 27, this must also be an eigenvector with eigenvalue 27. And it is indeed orthogonal to (1,-2,-2,0). (This was built into the construction, but it doesn't hurt to check the arithmetic at the end.)

6. a) A represents adding 3. B represents a clockwise rotation of $\pi/4$ radians, or multiplication by $e^{-i\pi/4}$.

b) $ABz - BAz = (ze^{-i\pi/4}+3)-(z+3)e^{-i\pi/4}) = 3(1-e^{-i\pi/4}) = 3(1-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})$.

Extra Credit. C takes everything in the complex plane to a single point. Since the z dropped out of the calculation in 6b), this is *not* adding or multiplying by $3(1-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})$, but rather changing all values directly into this one.