Problem Set 6
Physics 104A

Due November 3, 2009 at start of class; no late homework accepted

Primary topic: calculating Fourier expansions
Secondary topic: more linear algebra

Do Boas 7.5.5, 7.9.3, and 7.9.18, plus the following.

1. Let A be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_n$. Show that $\lambda_1 \le \langle v|A|v\rangle \le \lambda_n$ for all unit vectors $|v\rangle$.

2. a) Let U be a unitary matrix with eigenvectors $|v\rangle$ and eigenvalues λ. Act $U^\dagger$ on the equation $U|v\rangle = \lambda|v\rangle$ to show that $U^\dagger$ has the same eigenvectors as U. What are the corresponding eigenvalues?

b) Evaluate $\langle Uv|v\rangle \equiv (Uv)^\dagger v$ in two different ways to find a condition on the values λ. What is the corresponding condition for orthogonal matrices?

3. a) Do Boas 7.8.12a.

b) Do Boas 7.8.12b.

c) In your two expansions from part a), compare the coefficients of $\cos nx$ and $\sin nx$ with those of e^{inx} and e^{-inx}. Show that the two expansions are equal to each other.

4. a) Find the exponential Fourier series on $[-1, 1]$ for $g(x) = x$.

b) Find the exponential Fourier series on $[-1, 1]$ for $h(x) = |x|$.

c) Find the exponential Fourier series on $[-1, 1]$ for the function $f(x) = 0(x < 0), x(x \ge 0)$ in two ways. First, try to write down the series using your answers to a) and b). Then check your answer by integrating directly.

d) After you're sure you understand part c), use your expansions for g and h to write down the Fourier series of $q(x) = -x/2(x < 0), 3x(x \ge 0)$ on $[-1, 1]$. (No integrals needed for this part!)

And a couple of typical test problems that need not be handed in:

T1. (20 minutes)

a) State whether each of the matrices A, B, C below is diagonal, symmetric, antisymmetric, orthogonal, Hermitian, or unitary. (A given matrix may have more than one of these properties; list all that apply.)

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) Find the eigenvalues and eigenvectors of A. Write down a matrix S that diagonalizes A. (S need not be orthogonal or unitary.) What is the resulting diagonal matrix, $S^{-1}AS$?

T2. (10 minutes)

Let

$$f(x) = \begin{cases} e^{i2x} & |x| < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |x| < \pi \end{cases}$$

Expand f in an exponential Fourier series on the interval $[-\pi, \pi]$. Your final answer for the coefficients should involve only real numbers.