

## Problem Set 5, Physics 104A

Due Tuesday October 27, 2009 at start of class

Late homework accepted until start of class Thursday October 29 for half credit

Primary topic: geometry of eigenvectors and linear transformations

1. Consider (non-orthogonal) coordinates  $\hat{\mathbf{m}} = \hat{\mathbf{x}}$ ,  $\hat{\mathbf{n}} = (2\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{5}$ , where  $\hat{\mathbf{x}}, \hat{\mathbf{y}}$  are Cartesian coordinates.
  - a) Find the Cartesian coordinates for the points  $(m, n) = (1, 1) = \hat{\mathbf{m}} + \hat{\mathbf{n}}$  and  $(m, n) = (\sqrt{5}, 1/2)$ .
  - b) If we view matrices as coordinate changes, what is the matrix that changes a vector's coordinates in terms of  $\hat{\mathbf{m}}$  and  $\hat{\mathbf{n}}$  into its coordinates in terms of  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$ ?
  - c) If we instead view matrices as distorting the plane itself rather than simply renaming directions, what is the matrix that maps physical vectors  $\mathbf{V} = \hat{\mathbf{m}}$  and  $\mathbf{W} = \hat{\mathbf{n}}$  into  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$ , respectively? How is this matrix related to the one in part b)?
  - d) Find the area of the parallelogram defined by  $\hat{\mathbf{m}}$  and  $\hat{\mathbf{n}}$ , and explain how it relates to the determinant of the matrix of part c).

2.
  - a) Let  $A, B$  be operators with  $[A, B] = I$ , where  $I$  is the identity operator. Find  $[A^2, B]$ .
  - b) Let  $A = \partial/\partial x$ ,  $B = x_{op}$ . (Here  $B$  is an OPERATOR:  $B$  takes a function  $f(x)$  to another function  $g(x) = xf(x)$ .) Calculate  $[\partial^2/\partial x^2, x_{op}]$  by comparing  $\partial^2/\partial x^2(xf(x))$  and  $x\partial^2/\partial x^2 f(x)$ .
  - c) Verify that your answer from b) agrees with the general formula from part a).

3.
  - a) Find the eigenvalues and eigenfunctions of the operator  $\frac{d}{dx}$ . (i.e., the solutions of  $\frac{df}{dx} = \lambda f$ .)
  - b) Repeat for the operator  $x_{op} \frac{d}{dx}$ .

4. Let  $A = \begin{pmatrix} -1 & 7/2 & -1/2 \\ -2 & 9/2 & -1/2 \\ -2 & 7/2 & 1/2 \end{pmatrix}$ . The vectors  $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ ,  $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  are all eigenvectors of  $A$ .

- a) What are the eigenvalues?
- b) What is a matrix  $T$  that diagonalizes  $A$ ; i.e.,  $T^{-1}AT$  is diagonal? (Since  $A$  is not symmetric,  $T$  won't be orthogonal.) What is the resulting diagonal matrix?
- c) The vector  $\mathbf{v} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$  is also an eigenvector of  $A$ . What is the eigenvalue of  $\mathbf{v}$ ? Find an eigenvector of  $A$  which is perpendicular to  $\mathbf{v}$ .

5. For each matrix, find the eigenvalues and eigenvectors by hand (no computers please!). Then write down a matrix  $S$  such that  $S^{-1}AS$  (or  $S^{-1}BS$ ) is diagonal, and the resulting diagonal matrix.

a)  $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}$                       b)  $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

6. A matrix  $S$  diagonalizes *both*  $A$  and  $B$ . Show  $[A, B] = 0$ .

**T1.** (8 minutes)

Operator  $R$  rotates the plane by 90 degrees, taking the vector  $(x, y)$  to  $(y, -x)$ . Operator  $S$  stretches the plane by a factor of 2 vertically, taking  $(x, y)$  to  $(x, 2y)$ . Describe the effect on the plane of the commutator  $[R, S]$ .

**T2.** (12 minutes)

Let  $A = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & 3 \\ 4 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 \end{pmatrix}$ .

a) Find the eigenvalues of  $A$ .

b) Find two eigenvectors of  $A$  which are independent but not orthogonal.