Eigenvalues and eigenvectors of Hermitian matrices

(from class, October 27 2009)

A Hermitian matrix is one where the Hermitian conjugate equals the original matrix, $A^\dagger = A$. Hermitian matrices are a generalization of real symmetric matrices and have various analogous properties. One of the most important is that eigenvectors with different eigenvalues are orthogonal. As a result, for a Hermitian matrix it is always possible to find a complete set of eigenvectors that are mutually orthonormal. Here is the proof that the eigenvectors are orthogonal:

Let $\mathbf{v}$ and $\mathbf{w}$ be eigenvectors of A, with $A\mathbf{v} = \lambda_v \mathbf{v}$ and $A\mathbf{w} = \lambda_w \mathbf{w}$. Since matrix multiplication is associative, we can evaluate $\mathbf{w}^\dagger A\mathbf{v}$ in two ways: either start by calculating $A\mathbf{v}$ and later left-multiply by $\mathbf{w}^\dagger$, or else start by calculating $\mathbf{w}^\dagger A$ and later right-multiply by $\mathbf{v}$. That is,
1) $\mathbf{w}^\dagger A\mathbf{v} = \mathbf{w}^\dagger(\lambda_v \mathbf{v}) = \lambda_v(\mathbf{w}^\dagger \mathbf{v})$
and
2) $\mathbf{w}^\dagger A\mathbf{v} = (A^\dagger \mathbf{w})^\dagger \mathbf{v} = (A\mathbf{w})^\dagger \mathbf{v} = (\lambda_w \mathbf{w})^\dagger \mathbf{v} = \lambda_w^*(\mathbf{w}^\dagger \mathbf{v})$

Here the first step of 2) used the rule that $(AB)^\dagger = B^\dagger A^\dagger$ and the fact that $\mathbf{w}^{\dagger\dagger} = \mathbf{w}$. The second step is because A is Hermitian.

Since 1) and 2) were both starting with the same quantity, the calculation shows that $\lambda_v(\mathbf{w}^\dagger \mathbf{v}) = \lambda_w^*(\mathbf{w}^\dagger \mathbf{v})$. Furthermore, if we did the calculation using $\mathbf{w} = \mathbf{v}$, we would get $\lambda_v(\mathbf{v}^\dagger \mathbf{v}) = \lambda_v^*(\mathbf{v}^\dagger \mathbf{v})$, or $\lambda_v = \lambda_v^*$, which means the eigenvalue must be real. Since $\mathbf{v}$ was an arbitrary eigenvector of A, *all* the eigenvalues of a Hermitian matrix are real.

Returning to the case where $\mathbf{w}$ and $\mathbf{v}$ may be different, we find $\lambda_v(\mathbf{w}^\dagger \mathbf{v}) = \lambda_w(\mathbf{w}^\dagger \mathbf{v})$. There are two possibilities for satisfying the equation: either $\mathbf{w}^\dagger \mathbf{v} = 0$, in which case both sides vanish, or $\lambda_v = \lambda_w$. In words, either two eigenvectors are orthogonal or they must have the same eigenvalue.

Hermitian matrices are crucial in quantum mechanics, where they represent observable quantities. The eigenvalues of the matrix are the possible results from a measurement of the corresponding observable. Since the eigenvalues are always real, measurements of physical quantities always give real numbers, which is reassuring!