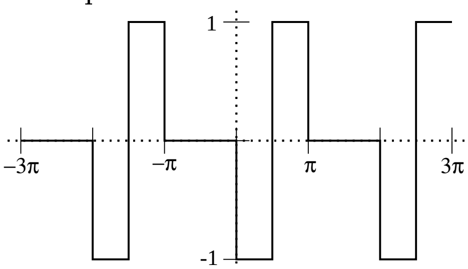
Answer Set 6
Physics 104A

Boas 7.5.5 $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi/2}(-1)dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} dx = 0$; $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx f(x)dx = -\frac{1}{\pi}\int_0^{\pi/2}\cos nx dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}\cos nx dx = -\frac{1}{n\pi}\sin nx|_0^{\pi/2} + \frac{1}{n\pi}\sin nx|_{\pi/2}^{\pi} = -2(-1)^{(n-1)/2}/n\pi$ for n odd and 0 for n even; $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx f(x)dx = -\frac{1}{\pi}\int_0^{\pi/2}\sin nx dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}\sin nx dx = \frac{1}{n\pi}\cos nx|_0^{\pi/2} - \frac{1}{n\pi}\cos nx|_{\pi/2}^{\pi} = -(2/n\pi)(1-(-1)^{n/2})$ for n even, 0 for n odd.

Putting these together gives $f(x) = -\frac{2}{\pi}(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - ... + \sin 2x + \frac{1}{3}\sin 6x + \frac{1}{5}\sin 10x + ...)$. Notice how the symmetry of the function leads to zeroes for half the cosine coefficients and three-quarters of the sine coefficients.



Boas 7.9.3 a) even: $-x^4 - 1$; odd: $x^5 + x^3$

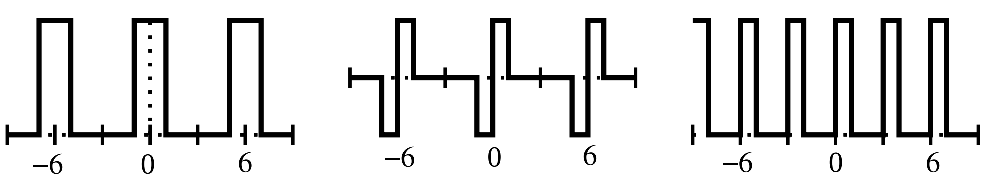
b) even: $1 + (e^x + e^{-x})/2$; odd: $(e^x - e^{-x})/2$

Boas 7.9.18 $f_c(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n}{3}x$, with $a_o = \frac{2}{3}\int_0^3 f(x)dx = \frac{2}{3}$ and for $n \neq 0$ $a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{\pi n}{3}x dx = \frac{2}{3}\int_0^1 \cos\frac{\pi n}{3}x dx = \frac{2}{3}(\frac{3}{\pi n}\sin\frac{\pi n}{3}x|_0^1) = \frac{2}{\pi n}\sin\frac{\pi n}{3}$.

$f_s(x) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi n}{3}x$, with $b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{\pi n}{3}x dx = \frac{2}{3}\int_0^1 \sin\frac{\pi n}{3}x dx = \frac{2}{3}(-\frac{3}{\pi n}\cos\frac{\pi n}{3}x|_0^1) = -\frac{2}{\pi n}(\cos\frac{\pi n}{3} - 1)$.

$f_p(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{2\pi n}{3}x + b_n \sin\frac{2\pi n}{3}x)$, with $a_o = \frac{2}{3}$ (exactly as in $f_c(x)$), and for $n \neq 0$ $a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{2\pi n}{3}x dx = \frac{2}{3}\int_0^1 \cos\frac{2\pi n}{3}x dx = \frac{2}{3}(\frac{3}{2\pi n}\sin\frac{2\pi n}{3}x|_0^1) = \frac{1}{\pi n}\sin\frac{2\pi n}{3}$ and $b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{2\pi n}{3}x dx = \frac{2}{3}\int_0^1 \sin\frac{2\pi n}{3}x dx = \frac{2}{3}(-\frac{3}{2\pi n}\cos\frac{2\pi n}{3}x|_0^1) = -\frac{1}{\pi n}(\cos\frac{2\pi n}{3} - 1)$.

even, period 6: odd, period 6: period 3:



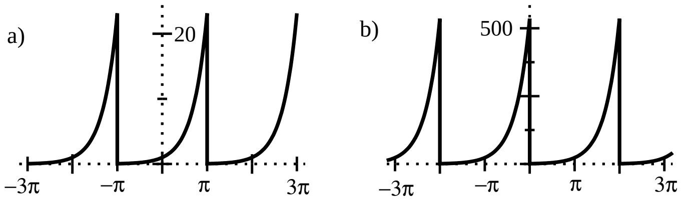
1. Since A is Hermitian, the eigenvalues are real and the eigenvectors can be chosen to be orthonormal. (If there are degenerate eigenvalues, it may also be possible to pick a non-orthogonal set of eigenvectors, but an orthonormal set is always an option.) Let $|v\rangle = \sum_{i=1}^{n}\alpha_i|v_i\rangle$, which must be possible since the $|v_n\rangle$'s are independent and hence a basis. Then $\langle v|A|v\rangle = \sum_{j=1}^{n}\sum_{i=1}^{n}\alpha_j^*\alpha_i\langle v_j|A|v_i\rangle = \sum_{j=1}^{n}\sum_{i=1}^{n}\alpha_j^*\alpha_i\langle v_j|\lambda_i|v_i\rangle = \sum_{j=1}^{n}\sum_{i=1}^{n}\alpha_j^*\alpha_i\lambda_i\langle v_j|v_i\rangle = \sum_{i=1}^{n}|\alpha_i|^2\lambda_i$. The last step used the orthonormality of the eigenvectors. Now, since $\lambda_1(\lambda_n)$ is the smallest (largest) eigenvalue, $\lambda_1 = \sum_{i=1}^{n}|\alpha_i|^2\lambda_1 \leq \sum_{i=1}^{n}|\alpha_i|^2\lambda_i \leq \sum_{i=1}^{n}|\alpha_i|^2\lambda_n = \lambda_n$. In the first and last steps, I used the fact that $|v\rangle$ is a unit vector: $1 = \langle v|v\rangle = \sum_{j=1}^{n}\sum_{i=1}^{n}\alpha_j^*\alpha_i\langle v_j|v_i\rangle = \sum_{i=1}^{n}|\alpha_i|^2$.

2. a) $U^\dagger U|v\rangle = U^\dagger\lambda|v\rangle$. Since $U^\dagger U = I$ and scalars commute with operators, this becomes $|v\rangle = \lambda U^\dagger|v\rangle$. Dividing by λ gives $\frac{1}{\lambda}|v\rangle = U^\dagger|v\rangle$, so $|v\rangle$ is an eigenvector of $U^\dagger$ with eigenvalue $\frac{1}{\lambda}$.

b) If $|v\rangle$ is a normalized eigenvector of U, then $\langle Uv|v\rangle = (Uv)^\dagger v = (\lambda v)^\dagger v = \lambda^*$. Alternatively, $(Uv)^\dagger v = v^\dagger U^\dagger v = v^\dagger(\frac{1}{\lambda}v) = \frac{1}{\lambda}$. (I have freely used commutativity of scalars and associativity.) This means $\lambda^* = \lambda^{-1}$, so $\lambda = e^{i\theta}$. The corresponding condition for real orthogonal matrices is that their real eigenvalues are ± 1. (Real orthogonal matrices are a special case of unitary matrices, so the general condition is again that $\lambda = e^{i\theta}$.)

3. a) For the exponential expansion, $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x-inx}dx = (e^{\pi(1-in)} - e^{-\pi(1-in)})/2\pi(1-in) = (-1)^n(e^\pi - e^{-\pi})/2\pi(1-in)$, so $e^x = (e^\pi - e^{-\pi})\sum_{-\infty}^{\infty}\frac{(-1)^n e^{inx}}{2\pi(1-in)}$ on $[-\pi,\pi]$.

For the sine-cosine expansion, $a_o = 2c_o = (e^\pi - e^{-\pi})/\pi$, $a_n = c_n + c_{-n} = \frac{(e^\pi-e^{-\pi})(-1)^n}{\pi(1+n^2)}$, and $b_n = i(c_n - c_{-n}) = -\frac{(e^\pi-e^{-\pi})(-1)^n n}{\pi(1+n^2)}$. I'll also do the cosine integral explicitly, since there's a bit of a trick to it: $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx dx = \frac{1}{\pi}[\frac{1}{n}e^x\sin nx|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}e^x \sin nx dx] = \frac{1}{\pi}[\frac{1}{n^2}e^x\cos nx|_{-\pi}^{\pi} - \frac{1}{n^2}\int_{-\pi}^{\pi} e^x\cos nx dx] = \frac{1}{\pi n^2}[e^\pi(-1)^n - e^{-\pi}(-1)^n] - \frac{1}{n^2}a_n = \frac{(-1)^n}{\pi n^2}(e^\pi - e^{-\pi}) - \frac{1}{n^2}a_n$. Solve this for a_n to get $a_n = \frac{(-1)^n}{\pi}(e^\pi - e^{-\pi})/(1+n^2)$.



b) $c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{x-inx}dx = (e^{2\pi(1-in)} - 1)/2\pi(1-in) = (e^{2\pi} - 1)/2\pi(1-in)$, so $e^x = (e^{2\pi} - 1)\sum_{-\infty}^{\infty}\frac{e^{inx}}{2\pi(1-in)}$ on $[0,2\pi]$.

The sine-cosine expansion has coefficients: $a_o = (e^{2\pi}-1)/\pi$; $a_n = \frac{(e^{2\pi}-1)}{\pi(1+n^2)}$; $b_n = -\frac{(e^{2\pi}-1)n}{\pi(1+n^2)}$.

c) (Clearly, part c) makes no sense if back in part a) you just got the sine-cosine expansion from the exponential expansion by algebra. If you did that, the "check" is exactly the calculation you did in the first place.) $a_o = 2c_o = (e^\pi - e^{-\pi})/\pi$, $a_n = c_n + c_{-n} = \frac{(e^\pi-e^{-\pi})(-1)^n}{\pi(1+n^2)}$, and $b_n = i(c_n - c_{-n}) = -\frac{(e^\pi-e^{-\pi})(-1)^n n}{\pi(1+n^2)}$

4. a) $g(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi nx}$. Here $c_o = \frac{1}{2}\int_{-1}^{1} x dx = 0$; and for $n \neq 0$, $c_n = \frac{1}{2}\int_{-1}^{1} e^{-i\pi nx}x dx = -\frac{1}{2i\pi n}xe^{-i\pi nx}|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1}\frac{1}{-i\pi n}e^{-i\pi nx}dx = \frac{i}{2\pi n}(e^{-i\pi n} - (-1)e^{i\pi n}) = \frac{i}{\pi n}(-1)^n$. (The integral left after the integration by parts vanishes.)

b) $h(x) = \sum_{n=-\infty}^{\infty} d_n e^{i\pi nx}$. Here $d_o = \frac{1}{2}\int_{-1}^{1}|x|dx = \frac{1}{2}[\int_{-1}^{0}(-x)dx + \int_0^1 x dx] = \frac{1}{2}$; and for $n \neq 0$, $d_n = \frac{1}{2}\int_{-1}^{1} e^{-i\pi nx}|x|dx = \frac{1}{2}[-\int_{-1}^{0} xe^{-i\pi nx}dx + \int_0^1 xe^{i\pi nx}dx] = \frac{1}{2}[\frac{x}{i\pi n}e^{-in\pi x}|_{-1}^{0} - \frac{1}{i\pi n}\int_{-1}^{0} e^{-i\pi nx}dx - \frac{x}{i\pi n}e^{-i\pi nx}|_0^1 + \frac{1}{i\pi n}\int_0^1 e^{-i\pi nx}dx] = \frac{1}{2}[\frac{1}{i\pi n}(0 + e^{i\pi n}) - \frac{1}{i\pi n}\frac{1}{-i\pi n}(1 - e^{i\pi n}) - \frac{1}{i\pi n}(e^{-i\pi n} - 0) + \frac{1}{i\pi n}\frac{1}{-i\pi n}(e^{-i\pi n} - 1)] = \frac{1}{i2\pi n}[(-1)^n + \frac{1}{i\pi n}(1-(-1)^n) - (-1)^n - \frac{1}{i\pi n}((-1)^n - 1)] = \frac{1}{\pi^2 n^2}[(-1)^n - 1] = -\frac{2}{\pi^2 n^2}$ (n odd), 0 (n even).

c) Since $f = (g+h)/2$, its Fourier series will be the average of the series for g and h. So, $f(x) = \frac{1}{4} + \sum_{n \text{ even}, n\neq 0}\frac{i}{2\pi n}e^{i\pi nx} - \sum_{n \text{ odd}}(\frac{i}{2\pi n} + \frac{1}{\pi^2 n^2})e^{i\pi nx}$. Doing the integrals directly gives coefficients $\frac{1}{2}\int_{-1}^{1} f(x)e^{-i\pi nx}dx = \frac{1}{2}\int_0^1 xe^{-i\pi nx}dx$. For $n = 0$ this gives $\frac{1}{4}$, and for $n \neq 0$ it gives $\frac{1}{2}[\frac{x}{-i\pi n}e^{-i\pi nx}|_0^1 - \int_0^1 \frac{1}{-i\pi n}e^{-i\pi nx}dx] = \frac{1}{2}[\frac{(-1)^n}{-i\pi n} - (\frac{1}{(-i\pi n)^2}e^{-i\pi nx}|_0^1)] = \frac{i(-1)^n}{2\pi n} + \frac{1}{2\pi^2 n^2}[(-1)^n - 1]$, which agrees with the previous answer.

d) $q(x) = \frac{5}{4}g(x) + \frac{7}{4}h(x) = \frac{7}{8} + \sum_{n \text{ even}, n\neq 0}\frac{5i}{4\pi n}e^{i\pi nx} - \sum_{n \text{ odd}}(\frac{5i}{4\pi n} + \frac{7}{2\pi^2 n^2})e^{i\pi nx}$.

T1. a) A: Hermitian, unitary, antisymmetric. B: Diagonal, symmetric. C: Orthogonal, unitary. (ANY real orthogonal matrix is also unitary.)

b) $\det\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = 0$ gives $\lambda^2 - 1 = 0$, so the eigenvalues are $\lambda = \pm 1$. The eigenvector corresponding to $\lambda = 1$ is $\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$, where α is some complex number. You can find α by going back to $\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = 0$. The first row of the matrix gives $-1 - i\alpha = 0$, so the eigenvector is $\begin{pmatrix} 1 \\ i \end{pmatrix}$. (This wouldn't work if the first entry of the eigenvector had in fact been 0, but you can easily check that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is not an eigenvector.) Similarly, the eigenvector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ corresponds to $\lambda = -1$.

The matrix S has the eigenvectors as its columns: $S = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. The corresponding diagonal matrix has the eigenvalues as its entries, in the same order that the eigenvectors appear in S: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (If you instead took $S = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, the diagonal matrix would be $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.)

T2. $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, with $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx f(x) e^{-inx} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx e^{i(2-n)x}$. For $n \neq 2$ this becomes $c_n = \frac{1}{2\pi}\frac{e^{i(2-n)x}}{i(2-n)}\Big|_{-\pi/2}^{\pi/2} = (\sin(2-n)\pi/2)/(2-n)\pi$. This is actually 0 for n even, and $(-1)^{\frac{n-1}{2}}/(2-n)\pi$ for n odd. The case $n = 2$ has to be done separately; $c_2 = 1/2$.