Answer Set 5, Physics 104A

1. a) $(1,1) \to (1+2/\sqrt{5}, 1/\sqrt{5})$; $(\sqrt{5}, 1/2) \to (\sqrt{5}+1/\sqrt{5}, 1/2\sqrt{5})$

b) $\begin{pmatrix} 1 & 2/\sqrt{5} \\ 0 & 1/\sqrt{5} \end{pmatrix}$

c) $\begin{pmatrix} 1 & -2 \\ 0 & \sqrt{5} \end{pmatrix}$; this is the inverse of the matrix in part b)

d) The area of the parallelogram is base $\times$ height $= 1/\sqrt{5}$. You could also get this from $|\hat{\mathbf{m}} \times \hat{\mathbf{n}}|$. The determinant of the matrix in c) is $\sqrt{5}$, which means that the transformation increases areas by $\sqrt{5}$. This is consistent, since the parallelogram gets mapped to the square defined by $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, which has area $\frac{1}{\sqrt{5}}(\sqrt{5}) = 1$.

2. a) $[A^2, B] = (AA)B - B(AA) = A(AB) - (BA)A = A(I + BA) - BAA = A + ABA - BAA = A + (I + BA)A - BAA = 2A$. (Everything associates, so I dropped those parentheses after the first two steps.)

b) $[\frac{\partial^2}{\partial x^2}, x]f = \frac{\partial^2}{\partial x^2}(xf) - x\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(f + x\frac{\partial f}{\partial x}) - x\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} - x\frac{\partial^2 f}{\partial x^2} = 2\frac{\partial f}{\partial x}$. Since this holds for ANY function f, $[\frac{\partial^2}{\partial x^2}, x] = 2\frac{\partial}{\partial x}$.

c) $[\frac{\partial}{\partial x}, x]f = \frac{\partial}{\partial x}(xf) - x\frac{\partial}{\partial x}f = (f + x\frac{\partial}{\partial x}f) - x\frac{\partial}{\partial x}f = f$, so $[\frac{\partial}{\partial x}, x] = I$. Then part a) shows that $[\frac{\partial^2}{\partial x^2}, x] = 2\frac{\partial}{\partial x}$, just as found in b).

3. a) The eigenfunction corresponding to eigenvalue λ is $f_\lambda(x) = Ce^{\lambda x}$. All complex numbers are eigenvalues, and C, the integration constant, can also be any complex number.

b) The eigenfunction corresponding to eigenvalue λ is $f_\lambda(x) = Ax^\lambda$. Again, any complex number is an eigenvalue. (To raise x to a complex power, note that $x^\lambda = e^{\ln(x^\lambda)} = e^{\lambda \ln x}$, and we know how to deal with complex powers of e.)

4. a) To get the eigenvalues, plug in the given vectors to $Av = \lambda v$. Actually all you need to do is find one entry of Av and see how it relates to the same entry of v. For example, say we calculate the last entry of Av for the first eigenvector: -2(1)+7/2(1)+1/2(3)=3, so the eigenvalue is 1. The second and third eigenvectors give eigenvalues of 1 and 2, respectively.

b) One possibility is $T = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$, so the resulting diagonal matrix is $T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, with the diagonal entries being the eigenvalues corresponding to the columns of T. Note that you can write down the diagonal matrix without actually computing T^{-1}, and that you needn't normalize the columns of T.

c) I'm once again going to save space by writing everything as row vectors. Since (1 1 3) and (-2 -1 1) both have eigenvalue 1, *any* linear combination of them also is an eigenvector with eigenvalue 1. To find an eigenvector perpendicular to $\mathbf{v}$, use a Gram-Schmidt procedure beginning with $\mathbf{v}$ and $\mathbf{w}$ =(1 1 3). (Any vector in the plane with eigenvalue 1 can be written as a linear combination of these two. We're going to find the component of $\mathbf{w}$ in the direction of $\mathbf{v}$, then subtract that component from $\mathbf{w}$ to find a vector in the $\lambda = 1$ plane which is perpendicular to $\mathbf{v}$.) First normalize $\mathbf{v}$: $\hat{\mathbf{v}} = 1/\sqrt{17}$(-1 0 4). The component of $\mathbf{w}$ parallel to $\hat{\mathbf{v}}$ is $(\hat{\mathbf{v}} \cdot \mathbf{w})\hat{\mathbf{v}} = 11/17$(-1 0 4). Subtracting this from $\mathbf{w}$ gives (28/17 1 7/17), an eigenvector of A with eigenvalue 1.

5. a) The second column doesn't "talk" to either of the others, so one eigenvector is (0,1,0), with eigenvalue 2. The corner entries form the 2×2 matrix $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$. You can find the eigenvectors and eigenvalues of this in various ways. One way is just to call an eigenvector (1,β) with corresponding eigenvalue λ, write out the two equations that result from $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \beta \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ \beta \end{pmatrix}$, and solve for λ and β. The answers are (1,-2) with eigenvalue -2 and (1,$\frac{1}{2}$) with eigenvalue 3; the corresponding eigenvectors of the original matrix are (1,0,-2) and (1,0,$\frac{1}{2}$). (As a check, note that the product of the eigenvalues equals the determinant of the original matrix, and that the sum of the eigenvalues equals the trace of the original matrix.) For $S = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -2 & \frac{1}{2} \end{pmatrix}$, the diagonal matrix is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

b) By inspection, (0,-1,1) has eigenvalue -2 and (1,-1,-1) has eigenvalue -1. I know that the sum of the eigenvalues must equal the sum of the diagonal of the original matrix, so the third eigenvalue must be 2. The corresponding eigenvector is (2,1,1), which you can get by inspection or by writing it as (x,y,1) and plugging into the eigenvalue equation with $\lambda = 2$. For $S = \begin{pmatrix} 0 & 1 & 2 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, the diagonal matrix is $\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

6. A matrix S diagonalizes *both* A and B. Show $[A, B] = 0$. $A_D = S^{-1}AS$ and $B_D = S^{-1}BS$. Multiply each equation by S on the left and S^{-1} on the right to get $A = SA_DS^{-1}$, $B = SB_DS^{-1}$. Then $[A, B] = (SA_DS^{-1})(SB_DS^{-1}) - (SB_DS^{-1})(SA_DS^{-1}) = SA_DB_DS^{-1} - SB_DA_DS^{-1} = S(A_DB_D - B_DA_D)S^{-1}$. This vanishes since any two diagonal matrices commute; their product is a diagonal matrix with each entry the product of the corresponding diagonal entries.

T1. The commutator is $[R, S] = RS - SR$. The operator RS means first act S, then R. So $RS(x, y) = R(x, 2y) = (2y, -x)$. Similarly $SR(x, y) = S(y, -x) = (y, -2x)$. So $[R, S](x, y) = (2y, -x) - (y, -2x) = (y, x)$. This is a reflection, flipping the plane about the line $y = x$.

T2. a) The matrix is essentially two 2x2 matrices. The matrix $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$ corresponds to the first and third rows and columns, and $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ to the second and fourth rows and columns. Any eigenvector (α, β) of the first of these smaller matrices corresponds to an eigenvector (α, 0, β, 0) of the 4x4 matrix, with the same eigenvalue; and similarly for the second 2x2 matrix. By inspection, the eigenvalues then are 5, -3, 5, and -1. (Or, you could find the roots of the characteristic polynomials of the 2x2 matrices.)

b) Since the matrix is symmetric, eigenvectors belonging to distinct eigenvalues are orthogonal. However, the eigenvalue 5 is repeated, so there is an entire plane of vectors with this eigenvalue. The two eigenvectors obtained from the 2x2 matrices are (1, 0, 1, 0) and (0, 1, 0, 1), which are orthogonal, but adding them together gives another eigenvector which is not orthogonal to either one. So one pair of independent, non-orthogonal eigenvectors is (1, 0, 1, 0) and (1, 1, 1, 1).