Answer Set 4
Physics 104A

Boas 3.5.30 The normal to the plane is (3,-2,-6), and a vector ending on the plane is (3,1,0). The angle between these is $\cos\theta = 7/\sqrt{(49)(10)} = 1/\sqrt{10}$. The distance from the origin to the plane is the length of (3,1,0) times $\cos\theta$, which equals 1.

Boas 3.5.35 The vector (2,-3,6) is in the direction of the line. Find the angle between it and (3,4,-1)-(2,5,1)=(1,-1,-2): $\cos\theta = -7/\sqrt{(49)(6)} = -1/\sqrt{6}$. The distance to the line is the length of (1,-1,-2) times $|\sin\theta|$, or $\sqrt{5}$.

Boas 3.7.7 Definite integration is a linear operator. (Unlike the derivative operator, discussed in class, which takes one function to another function, the definite integral takes a function to a number.) The linearity conditions are $\int_0^1 (f(x)+g(x))dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$ and $\int_0^1 (\lambda f(x))dx = \lambda \int_0^1 f(x)dx$.

Boas 3.7.10 The reciprocal is not linear. For example: Acting on numbers: $1/2 + 1/3 \neq 1/5$. Acting on functions: if $f(x) = x$ and $g(x) = x$, then $1/f + 1/g = 2/x \neq 1/(f+g) = 1/(2x)$.

1. I'll post sample C++ programs on the course web page, although I rarely find reading someone else's code useful. For $\sum_{n=0}^{\infty}(0.1)^n$, the exact value is $\frac{1}{1-0.1} = 1.11111...$, and the sum is closer than 1% for $N = 2$. For $\sum_{n=0}^{\infty}(0.9)^n$, the exact value is $\frac{1}{1-0.9} = 10$, and the sum is closer than 1% for $N = 43$. Starting at $n = 1$ doesn't change the number of terms needed, so it increases N by one. The sums become $\sum_{n=1}^{\infty}(0.1)^n = 0.111111...$ and $\sum_{n=1}^{\infty}(0.9)^n = 9$.

2. a) Convergence is not uniform; it gets slower and slower near $x = \sqrt{2}$.

b) Convergence is not uniform; it gets arbitrarily slow near $x = 1$.

c) Convergence is uniform. Everything goes at least as fast as $x = 1$.

d) Convergence not uniform; it gets arbitrarily slow as $x \to \infty$.

e) Convergence gets arbitrarily slow for x near 1 or -1, so it is not uniform.

3. a) 1

b) $5\hat{\mathbf{z}}$

c) $\cos\theta = |\mathbf{v}\cdot\mathbf{w}|/vw = 1/\sqrt{13}\sqrt{2} = .196$, so $\theta = 79$ degrees

d) $\frac{2}{\sqrt{13}}\hat{\mathbf{x}} + \frac{3}{\sqrt{13}}\hat{\mathbf{y}}$

4. $(a,b,c)\cdot(a,b,c) = a^2+b^2+c^2$ is generally larger than $(a,b,c)\cdot(b,c,a)$. The vectors (a,b,c) and (b,c,a) both have the same magnitude, $\sqrt{a^2+b^2+c^2}$. Then $(a,b,c)\cdot(b,c,a) = (a^2+b^2+c^2)\cos\theta$, and $\cos\theta < 1$ unless $a = b = c$. (If $a = b = c$ the two scalar products are equal.)

5. a) (I'm going to write all vectors as row vectors for this problem, just to save space.) Normalize (1,1,0) to get the first basis vector, $\mathbf{v_1} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$. Dotting (1,1,-1) with $\mathbf{v_1}$ gives $\sqrt{2}$. Now take $(1,1,-1) - \sqrt{2}\mathbf{v_1} = (0,0,-1)$. In general we'd have to normalize this vector to get $\mathbf{v_2}$, but here we're in luck because it's already normalized. So $\mathbf{v_2} = (0,0,-1)$. Finally, take the dot product of (3,0,4) with each of $\mathbf{v_1}$ and $\mathbf{v_2}$ and subtract off appropriate multiples of these basis vectors: $(3,0,4) - 3/\sqrt{2}\mathbf{v_1} - (-4)\mathbf{v_2} = (3/2, -3/2, 0)$. Normalizing this gives $\mathbf{v_3} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$.

b) $< F_1|F_1 >= 1$, so $g_1(x) = F_1(x) = 1$. $< F_2|g_1 >= \int_0^1 x dx = \frac{1}{2}$. (Note that the odd/even argument from class does not show F_1, F_2 are not orthogonal, since the interval [0,1] is not symmetric around 0.) Then $s_2(x) = F_2(x) - \frac{1}{2}g_1(x) = x - \frac{1}{2}$. Normalizing gives $g_2(x) = \sqrt{3}(2x-1)$. $< F_3|g_1 >= \int_0^1 e^{-x}dx = 1-\frac{1}{e}$ and $< F_3|g_2 >= \int_0^1 \sqrt{3}(2x-1)e^{-x}dx = \sqrt{3}(2x-1)(-e^{-x})|_0^1 - \int_0^1 \sqrt{3}(2)(-e^{-x})dx = \sqrt{3}(-2x-1)e^{-x}|_0^1 = -3\sqrt{3}/e + \sqrt{3}$. So $s_3(x) = e^{-x} - (1-\frac{1}{e}) - \sqrt{3}(1-\frac{3}{e})\sqrt{3}(2x-1) = e^{-x} + 2 - \frac{8}{e} + 6(\frac{3}{e}-1)x$. Normalize to find $g_3(x) = (e^{-x} + 2 - \frac{8}{e} + 6(\frac{3}{e}-1)x)/\sqrt{\frac{20}{e} - \frac{28.5}{e^2} - 3.5}$. (The final normalization step is very ugly, but you should have gotten as far as $s_3(x)$ in the calculation.) The "angle" between F_1 and F_2 has $\cos\theta =< F_1|F_2 > /\sqrt{< F_1|F_1 >< F_2|F_2 >} = \sqrt{3}/2$, so θ is $\pi/6$ radians.

c) First note that the coefficient of g_3 must be $3\sqrt{\frac{20}{e} - \frac{28.5}{e^2} - 3.5}$, to get the exponential part of h correct. But g_3 also introduces a linear term, which we can get rid of by subtracting $3\sqrt{3}(\frac{3}{e}-1)g_2(x)$. These factors of g_2 and g_3 together introduce a constant term $3(2-\frac{8}{e})+9(\frac{3}{e}-1) = \frac{3}{e}-3$, so add $(1-\frac{3}{e})g_1(x)$. All together, $h(x) = (1-\frac{3}{e})g_1(x) + 3\sqrt{3}(1-\frac{3}{e})g_2(x) + 3\sqrt{\frac{20}{e} - \frac{28.5}{e^2} - 3.5}g_3(x)$.

T1. $\mathbf{v} \cdot \mathbf{v} = \sum_{n=0}^{\infty} \frac{1}{3^{2n}} = 1/(1-\frac{1}{9}) = 9/8$.

T2. **a)** $\int_0^2 [f(x)]^2 dx = \int_0^2 x dx = \frac{x^2}{2}|_0^2 = 2$, so the normalized function is $f(x)/\sqrt{2} = \sqrt{x/2}$.

b) Start with some function that isn't a multiple of $f(x)$. I'll use $g(x) = 1$, since it's a nice simple function. The component of g in the direction of f has length $\int_0^2 g(x)\sqrt{x/2}dx = \int_0^2 \sqrt{x/2}dx = \frac{2}{3\sqrt{2}}x^{3/2}|_0^2 = 4/3$. Subtract off this component to get the function $h(x) = 1 - \frac{4}{3}\sqrt{x/2}$, which is orthogonal to f. (You can check that $\int_0^2 h(x)f(x)dx = 0$.)