Answer Set 3
Physics 104A

1. $n^{10} < 53,000,000 \times 5^n < 10^n < \sqrt{n!/2} < n!$ for large n. Constant prefactors, even one as big as 53,000,000, don't matter in the large-n limit. (You can figure out the ordering by looking at the ratio of the $(n+1)$-term and the n-term for each expression. The expressions that grow the fastest must eventually be the biggest.)

2. a) Diverges, by comparison with $1/n$: $\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$.

b) Diverges, since the terms get bigger and bigger. Each term is the previous term times $n/10$, which is larger than one for $n > 10$.

c) Diverges, by comparison with $1/n$. Note that $\ln n^{1/n} = \frac{1}{n} \ln n < 1$. Exponentiating both sides gives $n^{1/n} < e$. So, $\sum_{n=1}^{\infty} \frac{1}{nn^{1/n}} > \sum_{n=1}^{\infty} \frac{1}{ne}$.

d) Converges, by comparison with $1/n^2$. For large n, $n(n-1)$ looks a lot like n^2. In fact, $n(n-1) > n^2/2$ for $n \geq 2$. So, $\sum_{n=1}^{\infty} \frac{1}{n(n-1)} < \sum_{n=1}^{\infty} \frac{1}{n^2/2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$.

e) Diverges, by the integral test: $\sum_{n=2}^{\infty} \frac{1}{n \ln n} \geq \int_2^{\infty} \frac{dx}{x \ln x} = \ln(\ln x)|_2^{\infty} = \infty$.

f) Converges, by comparison with $1/2^n$: $\sum_{n=1}^{\infty} \frac{1}{n2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n}$.

3. The alternating series converges absolutely whenever the original all-positive series (i.e., the series in problem 2) converges. If the original series diverges, then the alternating series still converges conditionally as long as the terms steadily decrease in magnitude.

a) Converges conditionally.

b) Since the terms are growing larger and larger, this diverges.

c) Converges conditionally.

d) Converges absolutely.

e) Converges conditionally.

f) Converges absolutely.

4. a) $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n} = \sum_{n=0}^{\infty} (x^2/2)^n$ converges for $|x^2/2| < 1$, or $|x| < \sqrt{2}$. Note that if you try to use the ratio test on this series, you need to use a modified version that accounts for the fact that coefficients of odd terms vanish. The terms are never negative, so there is never conditional convergence.

b) Converges for $|x| < 1$ (comparison with $\sum (x^3)^n$) and diverges for $|x| > 1$ (comparison with $\sum 1/n$). Also converges (conditionally) at $x = -1$, where the series becomes the alternating harmonic series.

c) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges for $|x| \leq 1$, by comparison with $\sum 1/n^2$. The series diverges for $|x| > 1$ since the terms become increasing for large n. Convergence is never conditional.

d) $\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$ converges for all x. Comparison with $\sum 1/2^n$ works, for the large-n tail of the series. Convergence is never conditional.

e) $\sum_{n=0}^{\infty} \frac{1}{1+x^n}$ converges for $x < -1$ (alternating series with decreasing terms) and for $x > 1$ (by comparison with $\sum 1/x^n$). Also no conditional convergence.

5. The ugly way is expanding first in x and then in y. Expanding in y, then x (or even better, y^2 and then x) is less bad. I'll use a third route here, expanding $g(z) = \sqrt{1+z}$ and then setting $z = x+y^2$. The first few terms are $g(z) \approx 1+\frac{1}{2}z-\frac{1}{8}z^2+\frac{1}{16}z^3 \approx 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{2}y^2+\frac{1}{16}x^3-\frac{1}{4}xy^2$.

6. a) If $f(z) = \frac{1}{z+i}$, then $f'(z) = \frac{-1}{(z+i)^2}, f''(z) = \frac{2}{(z+i)^3}, f'''(z) = \frac{-6}{(z+i)^4}$. This gives $\frac{1}{z+i} = -i + z + iz^2 - z^3 + ... = \frac{1}{2+i} - \frac{z-2}{(2+i)^2} + \frac{(z-2)^2}{(2+i)^3} - \frac{(z-2)^3}{(2+i)^4} + ...$

The expansion about zero converges for $|z| < 1$ (radius of convergence is 1); the expansion about $z = 2$ converges on a circle of radius $\sqrt{5}$ centered at $z = 2$. ($\sqrt{5}$ is the distance between $z = 2$ and the singularity of f at $z = -i$.) The $z \approx 2$ expansion converges at $z = 0$ but not vice versa.

b) $(z-2)^2 = z^2 - 4z + 4; (z-2)^3 = z^3 - 6z^2 + 12z - 8$. The constant terms in the $z \approx 2$ expansion are $\frac{1}{2+i} + \frac{2}{(2+i)^2} + \frac{4}{(2+i)^3} + \frac{8}{(2+i)^4} + ...$, so the series we need is $\sum_{n=0}^{\infty} \frac{2^n}{(2+i)^{n+1}} = \frac{1}{2+i}\sum_{n=0}^{\infty}(\frac{2}{2+i})^n = \frac{1}{2+i}\frac{1}{1-2/(2+i)} = \frac{1}{i} = -i$, which is exactly the constant term in the $z \approx 0$ expansion. We could do similar tricks at other points, but the series we have to sum to get from one Taylor expansion to the other only converge within the radius of convergence of the Taylor expansion.

7. Yes, what's left converges! One hint is that, for very very large denominators, almost ALL the terms get thrown out. (E.g., an overwhelming majority of 1000-digit numbers contain at least one 7.) In fact, there are 8 allowed 1-digit denominators. There are 72 allowed 2-digit denominators (8 possibilities for the first digit, since neither 0 nor 7 is allowed, and 9 possibilities for the second digit). There are 8×9^2 allowed 3-digit denominators (again, 8 choices for the first digit and 9 for each subsequent digit). In general, there are $8 \times 9^{n-1}$ n-digit denominators. Now replace each n-digit denominator by the smallest n-digit denominator (i.e., the largest fraction). This gives 8 1's, 72 $\frac{1}{10}$'s, etc. The comparison test says the original series is less than $(1 \times 8) + (\frac{1}{10} \times 8 \times 9) + (\frac{1}{100} \times 8 \times 9^2) + (\frac{1}{1000} \times 8 \times 9^3) + ... = 8(1 + \frac{9}{10} + (\frac{9}{10})^2 + (\frac{9}{10})^3 + ...) = 8/(1 - \frac{9}{10}) = 80$. (Finding the upper bound of 80 for the sum is more than the question asked for.)

T1. a) If x is allowed to take any complex value, the resulting function has singularities in the complex plane at 3, $2i$, and $-2i$. The latter two are closest to the origin, so they determine the radius of convergence of an expansion around the origin. This radius is 2, and the expansion converges for real x with $-2 < x < 2$.

b) Factor $x^3/4$ out of the summation to get $\frac{x^3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2+n}$. For large n the terms of the sum are basically $1/n^2$; in fact, they're always a bit smaller than that. By comparison, since $\sum \frac{1}{n^2}$ converges, the sum in this problem does too, for ALL x. As far as the summation goes here, x is a constant. It affects the final value but not whether the series converges.