Answer Set 2, Physics 104A

1. a) Use $\delta(2x-3) = \frac{1}{2}\delta(x-\frac{3}{2})$, so the integral is $\frac{1}{2}f(\frac{3}{2})$.

b) The integral equals 0. (Just plug in $z = \pi$.)

c) $g(x) = 2x^2 - x - 1 = (2x+1)(x-1)$ vanishes at $x = -1/2$ and $x = 1$. Only the latter is in the range of integration, and $g\prime(1) = 3$. The integral is then $f(1)/|g\prime(1)| = f(1)/3$.

d) $x^3 - \pi^2 x$ vanishes as $x = -\pi, 0, \pi$. Its derivatives at these points are $2\pi^2, -\pi^2, 2\pi^2$, respectively. Adding each root's contribution, the integral equals $-\frac{1}{2\pi^2} + \frac{1}{\pi^2} - \frac{1}{2\pi^2} = 0$.

e) This one doesn't exist; the integral is infinite. The problem is that at $x = 0$, both $g(x) = x^3$ and $g'(x) = 3x^2$ vanish.

f) The zeroes of $g(x) = e^\theta \cos\theta$ within the integration range are $\theta = \pi/2, 3\pi/2$. At these points $|g'(x)| = e^{\pi/2}, e^{3\pi/2}$. The integral becomes $e^{-\pi/2} - e^{-3\pi/2}$.

g) The solutions of $2x^2 + 2x + 1 = 0$ are complex, so there are no zeroes on the integration path and the result is zero.

h) The integral vanishes for $r < 2$, since its range contains no points where the delta function vanishes (i.e., where $r^2 - x^2 = 4$). For $r > 2$, take $g(x) = \sqrt{r^2 - x^2} - 2$, so $|\frac{dg}{dx}| = \frac{x}{\sqrt{r^2-x^2}}$. At the zero of g this becomes $|\frac{dg}{dx}| = \frac{\sqrt{r^2-4}}{2}$. So, $\int f(x)\delta(\sqrt{r^2-x^2}-2)dx = \frac{2}{\sqrt{r^2-4}} f(\sqrt{r^2-4})$.

2. a) $-f'(1)$

b) Zeroes of $\sin x$ between $\pm 5\pi/2$ are at $-2\pi, -\pi, 0, \pi$, and 2π. At each point the absolute value of the derivative is 1. Add $-(e^{-2x})'$ at each zero to get $2e^{-4\pi} + 2e^{-2\pi} + 2 + 2e^{2\pi} + 2e^{4\pi}$.

c) We can evaluate the three integrals in any order. For example: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz\, f(x,y,z)\delta(\frac{x}{2}+1)\delta(y-3)\delta(\sqrt{6}z+1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dydz\, 2f(-2,y,z)\delta(y-3)\delta(\sqrt{6}z+1) = \int_{-\infty}^{\infty} dz\, 2f(-2,3,z)\delta(\sqrt{6}z+1) = 2/\sqrt{6} f(-2,3,-1/\sqrt{6})$.

d) Do integrals in either order. The x-integral give a $1/3$ factor from the $3x$ in the δ-function's argument. Plugging in $x = 5/3$ and $y = 0$ gives $\frac{25}{9} + 1 = 34/27$.

e) $\int_0^\infty dx \int_0^\infty dy \frac{\delta(x-y)}{(4x^2+5y^2+1)^{3/2}} = \int_0^\infty dx \frac{1}{(9x^2+1)^{3/2}} = \int_0^{\pi/2} \frac{1}{3}\sec^2\theta d\theta \frac{1}{(\tan^2\theta+1)^{3/2}} = \int_0^{\pi/2} \frac{1}{3}\cos\theta d\theta = \frac{1}{3}$. The integral over the upper half-plane has exactly the same value, since the same part of the line $x = y$ is included. The integral over the whole plane gives $\frac{2}{3}$.

f) $\int_0^{\pi/4}\int_0^{\pi/4} dydx\delta(x-2y)\cos\sqrt{2xy} = \int_0^{\pi/8} dy \cos\sqrt{4y^2} = \int_0^{\pi/8}\cos 2y = \frac{1}{2}\sin\pi/4 = \sqrt{2}/4$. (Note that $x = 2y$ is never satisfied for $y > \pi/8$, so the upper integration bound for y is reduced to $\pi/8$.) Alternatively, $\int_0^{\pi/4}\int_0^{\pi/4} dxdy\delta(x-2y)\cos\sqrt{2xy} = \int_0^{\pi/4} dx \frac{1}{2}\cos\sqrt{x^2} = \frac{1}{2}\sin x|_0^{\pi/4} = \sqrt{2}/4$. (This time the $\frac{1}{2}$ came from the $2y$ in the δ-function.)

T1. a) Let $g(\theta) = (\theta - \frac{\pi}{4})\sin\theta$ be the argument of the delta function. $g(\theta)$ vanishes at two spots in the integration range: $\theta = \pi/4$ and $\theta = 0$. The derivative is $g'(\theta) = (\theta - \frac{\pi}{4})\cos\theta + \sin\theta$, so $g'(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$ and $g'(0) = -\pi/4$. The integral becomes $\cos(\pi/4)/|g'(\pi/4)| + \cos(0)/|g'(0)| = 1 + \frac{4}{\pi}$.

b) Two important ideas here are that you can get rid of ONE integral, but not both, with the delta function; and that you have to be careful about what points are in the range of integration. Say you do the x integral first. For this integral, y is a constant. If $y > 5$, then you can NEVER have $x = y$ in the range of integration for x, so the x integral vanishes. However, if $0 < y < 5$, then we evaluate the rest of the integrand, in this case the constant function, at $x = y$ to get 1 for the x integral. Then the y integral becomes $\int_0^5 dy = 5$.

You could also do the integrals in the other order. Here, you treat x as a constant in the delta function. But this time, since $0 < x < 5$, the zero of the argument will ALWAYS be in the integration range of y, and the y integral is 1. That leaves the x integral, $\int_0^5 dx = 5$.