**Answer Set 1**

**Physics 240A**

**A&M 1.1 a)** Divide the time \( t \) into \( N \) intervals \( dt \). The electron has no collision in a particular time interval \( dt \) with probability \( 1 - \frac{dt}{\tau} \). It has no collision in \( N \) independent intervals \( dt \) with probability \( (1 - \frac{dt}{\tau})^N \). Now take the limit as \( N \to \infty \): \( \lim_{N \to \infty} (1 - \frac{dt}{\tau})^N = \lim_{N \to \infty} (1 - \frac{t}{N\tau})^N = e^{-t/\tau} \).

b) Assume a collision occurs at \( t = 0 \). For the next collision to be between \( t \) and \( t + dt \), there must be no collision for time \( t \) and then there must BE a collision in the next interval \( dt \). Using a), this probability is simply \( e^{-t/\tau} dt / \tau \).

c) The previous collision was \( t \) seconds ago if there was no collision in the previous \( t \) seconds and there WAS a collision just before that. (Same argument as in b), in fact, and yielding same formula.) The mean time to the last collision is then \( \int_0^\infty xe^{-x/\tau} \frac{dx}{\tau} = \int_0^\infty xe^{-x/\tau} dx \). Using a), this probability is simply \( e^{-t/\tau} dt / \tau \).

d) Exactly the same integrals as in c), so same result.

e) The point is that for the probability of collision in different time intervals to be independent, it can’t matter when a particle last collided. That is, the average time to a collision MUST be the same no matter when the previous collision was; otherwise different times WOULD depend on each other.

Mathematically, the key idea is that the tail of the exponential function, renormalized, looks exactly the same as the original function. That is, at \( t = 0 \) consider an electron whose previous collision was at \( t = -t_0 \). Just after that collision, there was a probability distribution \( f(t) \) governing the probability of collision \( t \) seconds later. (Properly normalized, \( f(t) = \frac{1}{\tau} e^{-t/\tau} \).) At time \( t = 0 \), we have extra information: we KNOW that the particle didn’t collide between \( -t_0 \) and 0. However, we have no reason to change the relative probabilities for collisions times after \( t = 0 \). In other words, we construct a new probability distribution \( g(t) \) where there is 0 chance of a collision before \( t = 0 \). So \( g(t) \) is proportional to \( f(t_0 + t) = \frac{1}{\tau} e^{-t_0+t/\tau} \) for \( t > 0 \), and 0 otherwise. Since \( g(t) \) is a probability distribution, \( \int_0^\infty g(t) dt = 1 \). The normalization gives back exactly the original distribution, \( g(t) = \frac{1}{\tau} e^{-t/\tau} \).

**A&M 1.2 a)** The energy just before the second collision is \( \frac{1}{2}mv^2 = \frac{1}{2}m(v_0 - \frac{E}{m}t)^2 = \frac{1}{2}m(v_0^2 - 2\frac{E}{m}v_0 \cdot \mathbf{E} + (\frac{E}{m}t)^2) \). Averaging over all possible \( v_0 \), the middle term vanishes. The first term equals the energy just after the second collision. This leaves the third term as the average energy loss, \( (\frac{eE}{m})^2 / 2m \).

b) Now do a weighted average over the time between collisions, \( \int_0^\infty \frac{(\frac{eE}{m})^2}{2m} e^{-t/\tau} dt / \tau = (eE\tau)^2 / m \). (Integrate by parts.) With \( n \) electrons per cm\(^3\) and each electron colliding every 1/\( \tau \) seconds on average, the energy loss per cm\(^3\) per second becomes \( n\frac{e^2}{m}E^2 = \sigma E^2 \). Multiply by \( AL \), the volume of the wire. With \( I = Aj = A\sigma E \) and \( R = \frac{L}{\sigma A} \), this becomes \( (AL)I^2 / \sigma A^2 = I^2R \).

**A&M 1.3** Consider the change in thermal energy of a particle which has \( t \) seconds between collisions and begins with velocity \( \mathbf{v} \). It travels a distance of \( \frac{1}{2}(-\mathbf{E}t)^2 + vt \). We need to find \( \varepsilon(T(r - \frac{1}{2}(-\mathbf{E}t)^2 + vt)) \). Taylor expanding, the energy difference becomes \( \frac{d\varepsilon}{dt} \cdot (\nabla T \cdot (-\frac{1}{2}(-\mathbf{E}t)^2 + vt)) \). In averaging over all directions of initial velocity the second term vanishes and the first term is unchanged. As in 2b), do a weighted average over the time between collisions. The integral is identical, with only prefactors changed,
so the average loss per electron per collision is 
\[ \frac{e\tau^2}{m} \frac{de}{dt} (\mathbf{E} \cdot \nabla T). \]
For \( n \) electrons with average time \( \tau \) between collisions, the total power loss (i.e., energy lost per time) is 
\[ \frac{n e \tau^2}{m} \frac{de}{dt} (\mathbf{E} \cdot \nabla T). \]
From an intuitive perspective, the \( \mathbf{E} : \nabla T \) term comes in because the energy loss per collision depends in part on how far along the temperature gradient the particle travels between collisions. If \( \mathbf{E} \) is perpendicular to the gradient, it has no effect on this distance. On the other hand, if \( \mathbf{E} \) is parallel to the gradient, the acceleration from the electric field has maximal effect on changing the distance traveled along the gradient.

1. a) The electrons, on average, are moving in the \(-\hat{x}\) direction, so they feel a force in the \(-\hat{y}\) direction. They build up on one side of the sample, until the resulting electric field from the uneven electron distribution produces a force exactly cancelling the magnetic force. That is, 
\[ -\varepsilon \left( \frac{J}{ne} \right) B - eE_y = 0, \]
where \( E_y \) is the electric field component in the \( \hat{y} \) direction. Solving, 
\[ E_y = -\frac{JB}{ne}. \]

b) The electric field is constant, so the Hall voltage is simply \(-E_y w\), where \( w \) is the width of the sample in the \( y \) direction. The negative sign represents that if \( E_y \) is positive, the voltage would increase in the \(-\hat{y}\) direction. The applied current is \( Jw t \), where \( t \) is the thickness of the sample in the \( z \) direction. Dividing gives Hall resistance \( -\frac{E_y}{Jt} = \frac{B}{nect} \). This has the practical consequence that in measuring Hall resistance, thinner samples give bigger signals. Hall probes are a standard way of measuring magnetic fields.

c) High sensitivity means a large change in the measured quantity (i.e., the Hall voltage) for a fixed change in field. This means that the carrier density \( n \) should be small (a material property), and the sample should be thin (low \( t \), a geometric property). One factor limiting how small \( t \) can be is that very thin films are fragile, and one wouldn’t want one’s Hall probe to be destroyed too easily. Another issue is that \( n \) is a function of temperature, so you need to know how hot the film is to interpret the Hall measurement. However, thin films have high resistance, which means Joule heating \( (I^2 R) \) from the measuring current may heat the film and change \( n \). Reducing the current to avoid heating will also reduce sensitivity.

d) \( R_H = -\frac{1}{ne} \). In this classical picture, \( R_H \) is always negative. (N.B. All the above is in cgs units—\( n \) in cm\(^{-3} \), \( e \) in esu, and \( c \) in cm/s. In SI units, the \( c \)’s go away, giving \( R_H = -\frac{1}{ne} \) with \( n \) in m\(^{-3} \) and \( e \) in Coulombs.)